Miscellaneous Examples

Example 9 Find the value of $\sin^{-1}(\sin\frac{3\pi}{5})$

Solution We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}(\sin\frac{3\pi}{5}) = \frac{3\pi}{5}$

But $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1} x$

However $\sin(\frac{3\pi}{5}) = \sin(\pi - \frac{3\pi}{5}) = \sin\frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore $\sin^{-1}(\sin\frac{3\pi}{5}) = \sin^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$

Example 10 Show that $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Solution Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$

Therefore $\sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$

Now $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$ (Why?)

and $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{64}{289}} = \frac{15}{17}$

We have $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$= \frac{4}{5} \times \frac{15}{17} + \frac{3}{5} \times \frac{8}{17} = \frac{84}{85}$$

Therefore $x - y = \cos^{-1}\frac{84}{85}$

Hence $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Example 11 Show that $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Solution Let $\sin^{-1}\frac{12}{13}=x,\quad \cos^{-1}\frac{4}{5}=y, \tan^{-1}\frac{63}{16}=z$

Then $\quad \sin x=\frac{12}{13},\quad \cos y=\frac{4}{5},\quad \tan z=\frac{63}{16}$

Therefore $\quad \cos x=\frac{5}{13},\ \sin y=\frac{3}{5},\ \tan x=\frac{12}{5}$ and $\tan y=\frac{3}{4}$

We have $\quad \tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}=-\frac{63}{16}$

Hence $\quad \tan(x+y)=-\tan z$

i.e., $\quad \tan(x+y)=\tan(-z)$ or $\tan(x+y)=\tan(\pi-z)$

Therefore $\quad x+y=-z$ or $x+y=\pi-z$

Since $\quad x$, y and z are positive, $x+y\neq -z$ (Why?)

Hence $\quad x+y+z=\pi$ or $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

Example 12 Simplify $\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]$, if $\frac{a}{b}\tan x>-1$

Solution We have,

$$\tan^{-1}\left[\frac{a\cos x-b\sin x}{b\cos x+a\sin x}\right]=\tan^{-1}\left[\frac{\frac{a\cos x-b\sin x}{b\cos x}}{\frac{b\cos x+a\sin x}{b\cos x}}\right]=\tan^{-1}\left[\frac{\frac{a}{b}-\tan x}{1+\frac{a}{b}\tan x}\right]$$

$$=\tan^{-1}\frac{a}{b}-\tan^{-1}(\tan x)=\tan^{-1}\frac{a}{b}-x$$

Example 13 Solve $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

Solution We have $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

or $$\tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right) = \frac{\pi}{4}$$

i.e. $$\tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$$

Therefore $$\frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$$

or $$6x^2 + 5x - 1 = 0 \text{ i.e., } (6x - 1)(x + 1) = 0$$

which gives $$x = \frac{1}{6} \text{ or } x = -1.$$

Since $x = -1$ does not satisfy the equation, as the L.H.S. of the equation becomes negative, $x = \frac{1}{6}$ is the only solution of the given equation.

Miscellaneous Exercise on Chapter 2

Find the value of the following:

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

Prove that

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

4. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$

5. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

6. $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

7. $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

8. $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Prove that

9. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$, $x \in [0, 1]$

10. $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}$, $x \in \left(0, \frac{\pi}{4}\right)$

11. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x$, $-\frac{1}{\sqrt{2}} \le x \le 1$ [Hint: Put $x = \cos 2\theta$]

12. $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solve the following equations:

13. $2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

14. $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x > 0)$

15. $\sin(\tan^{-1}x)$, $|x| < 1$ is equal to

(A) $\frac{x}{\sqrt{1-x^2}}$ (B) $\frac{1}{\sqrt{1-x^2}}$ (C) $\frac{1}{\sqrt{1+x^2}}$ (D) $\frac{x}{\sqrt{1+x^2}}$

16. $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$, then x is equal to

(A) $0, \frac{1}{2}$ (B) $1, \frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$

17. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{3\pi}{4}$

Summary

- The domains and ranges (principal value branches) of inverse trigonometric functions are given in the following table:

Functions	**Domain**	**Range (Principal Value Branches)**
$y = \sin^{-1} x$	$[-1, 1]$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \text{cosec}^{-1} x$	$\mathbf{R} - (-1,1)$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$y = \sec^{-1} x$	$\mathbf{R} - (-1, 1)$	$[0, \pi] - \{\frac{\pi}{2}\}$
$y = \tan^{-1} x$	$\mathbf{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	$\mathbf{R}$	$(0, \pi)$

- $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. In fact $(\sin x)^{-1} = \frac{1}{\sin x}$ and similarly for other trigonometric functions.
- The value of an inverse trigonometric functions which lies in its principal value branch is called the *principal value* of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = \sin^{-1} x \Rightarrow x = \sin y$
- $x = \sin y \Rightarrow y = \sin^{-1} x$
- $\sin (\sin^{-1} x) = x$
- $\sin^{-1} (\sin x) = x$
- $\sin^{-1} \frac{1}{x} = \text{cosec}^{-1} x$
- $\cos^{-1} (-x) = \pi - \cos^{-1} x$
- $\cos^{-1} \frac{1}{x} = \sec^{-1} x$
- $\cot^{-1} (-x) = \pi - \cot^{-1} x$
- $\tan^{-1} \frac{1}{x} = \cot^{-1} x$
- $\sec^{-1} (-x) = \pi - \sec^{-1} x$

- $\sin^{-1}(-x) = -\sin^{-1} x$
- $\tan^{-1}(-x) = -\tan^{-1} x$
- $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$
- $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$
- $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$
- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$
- $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$
- $2\tan^{-1} x = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2}$
- $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$
- $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), xy > 1;\ x, y > 0$

Historical Note

The study of trigonometry was first started in India. The ancient Indian Mathematicians, Aryabhata (476A.D.), Brahmagupta (598 A.D.), Bhaskara I (600 A.D.) and Bhaskara II (1114 A.D.) got important results of trigonometry. All this knowledge went from India to Arabia and then from there to Europe. The Greeks had also started the study of trigonometry but their approach was so clumsy that when the Indian approach became known, it was immediately adopted throughout the world.

In India, the predecessor of the modern trigonometric functions, known as the sine of an angle, and the introduction of the sine function represents one of the main contribution of the *siddhantas* (Sanskrit astronomical works) to mathematics.

Bhaskara I (about 600 A.D.) gave formulae to find the values of sine functions for angles more than 90°. A sixteenth century Malayalam work *Yuktibhasa* contains a proof for the expansion of sin (A + B). Exact expression for sines or cosines of 18°, 36°, 54°, 72°, etc., were given by Bhaskara II.

The symbols $\sin^{-1} x$, $\cos^{-1} x$, etc., for arc sin x, arc cos x, etc., were suggested by the astronomer Sir John F.W. Hersehel (1813) The name of Thales (about 600 B.C.) is invariably associated with height and distance problems. He is credited with the determination of the height of a great pyramid in Egypt by measuring shadows of the pyramid and an auxiliary staff (or gnomon) of known height, and comparing the ratios:

$$\frac{\mathrm{H}}{\mathrm{S}} = \frac{h}{s} = \tan \text{(sun's altitude)}$$

Thales is also said to have calculated the distance of a ship at sea through the proportionality of sides of similar triangles. Problems on height and distance using the similarity property are also found in ancient Indian works.