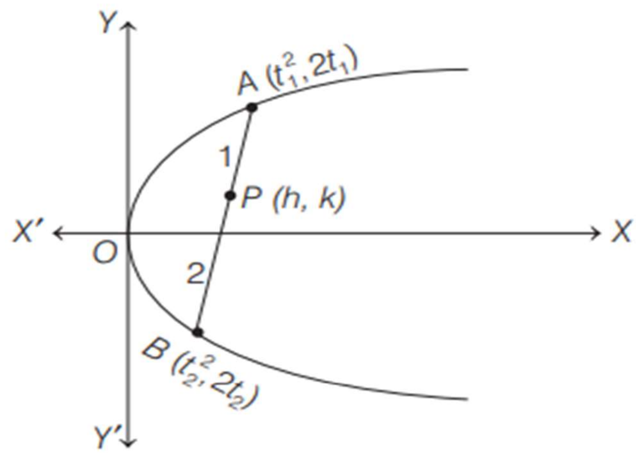
10. Show that the locus of a point that divides a chord of slope 2 of the parabola $y^2 = 4ax$ internally in the ratio 1 : 2 is a parabola. Find the vertex of this parabola. **(1995, 5M)**

Solution: -

10. Let $A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$ be coordinates of the end points of a chord of the parabola $y^2 = 4x$ having slope 2.

Now, slope of AB is

$$m = \frac{2t_2 - 2t_1}{t_2^2 - t_1^2} = \frac{2(t_2 - t_1)}{(t_2 - t_1)(t_2 + t_1)} = \frac{2}{t_2 + t_1}$$



But $m = 2$ [given]

$\Rightarrow$ $2 = \dfrac{2}{t_2 + t_1}$

$\Rightarrow$ $t_1 + t_2 = 1$ ...(i)

Let $P(h, k)$ be a point on AB such that, it divides AB internally in the ratio 1 : 2.

Then, $h = \dfrac{2t_1^2 + t_2^2}{2 + 1}$ and $k = \dfrac{2(2t_1) + 2t_2}{2 + 1}$

$\Rightarrow$ $3h = 2t_1^2 + t_2^2$ ...(ii)

and $3k = 4t_1 + 2t_2$ ...(iii)

On substituting value of t_1 from Eq. (i) in Eq. (iii)

$$3k = 4\,(1 - t_2) + 2t_2$$

$$\Rightarrow \qquad 3k = 4 - 2t_2$$

$$\Rightarrow \qquad t_2 = 2 - \frac{3k}{2} \qquad \ldots\text{(iv)}$$

On substituting $t_1 = 1 - t_2$ in Eq. (ii), we get

$$3h = 2\,(1 - t_2)^2 + t_2^2$$

$$= 2\,(1 - 2t_2 + t_2^2) + t_2^2$$

$$= 3t_2^2 - 4t_2 + 2 = 3\left(t_2^2 - \frac{4}{3}t_2 + \frac{2}{3}\right)$$

$$= 3\left[\left(t_2 - \frac{2}{3}\right)^2 + \frac{2}{3} - \frac{4}{9}\right] = 3\left(t_2 - \frac{2}{3}\right)^2 + \frac{2}{3}$$

$$\Rightarrow \qquad 3h - \frac{2}{3} = 3\left(t_2 - \frac{2}{3}\right)^2$$

$$\Rightarrow \qquad 3\left(h - \frac{2}{9}\right) = 3\left(2 - \frac{3k}{2} - \frac{2}{3}\right)^2 \qquad \text{[from Eq. (iv)]}$$

$$\Rightarrow \qquad 3\left(h - \frac{2}{9}\right) = 3\left(\frac{4}{3} - \frac{3k}{2}\right)^2$$

$$\Rightarrow \qquad \left(h - \frac{2}{9}\right) = \frac{9}{4}\left(k - \frac{8}{9}\right)^2$$

$$\Rightarrow \qquad \left(k - \frac{8}{9}\right)^2 = \frac{4}{9}\left(h - \frac{2}{9}\right)$$

On generalising, we get the required locus

$$\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right)$$

This represents a parabola with vertex at $\left(\frac{2}{9}, \frac{8}{9}\right)$.