5.4 Exponential and Logarithmic Functions

Till now we have learnt some aspects of different classes of functions like polynomial functions, rational functions and trigonometric functions. In this section, we shall learn about a new class of (related) functions called exponential functions and logarithmic functions. It needs to be emphasized that many statements made in this section are motivational and precise proofs of these are well beyond the scope of this text.

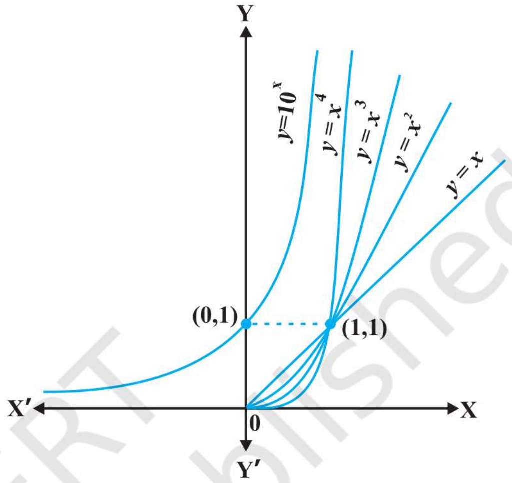

Fig 5.9

The Fig 5.9 gives a sketch of $y = f_1(x) = x$, $y = f_2(x) = x^2$, $y = f_3(x) = x^3$ and $y = f_4(x) = x^4$. Observe that the curves get steeper as the power of x increases. Steeper the curve, faster is the rate of growth. What this means is that for a fixed increment in the value of $x (> 1)$, the increment in the value of $y = f_n(x)$ increases as n increases for $n = 1, 2, 3, 4$. It is conceivable that such a statement is true for all positive values of n, where $f_n(x) = x^n$. Essentially, this means that the graph of $y = f_n(x)$ *leans* more towards the y-axis as n increases. For example, consider $f_{10}(x) = x^{10}$ and $f_{15}(x) = x^{15}$. If x increases from 1 to 2, f_{10} increases from 1 to 2^{10} whereas f_{15} increases from 1 to 2^{15}. Thus, for the same increment in x, f_{15} grow faster than f_{10}.

Upshot of the above discussion is that the growth of polynomial functions is dependent on the degree of the polynomial function – higher the degree, greater is the growth. The next natural question is: Is there a function which grows faster than any polynomial function. The answer is in affirmative and an example of such a function is

$$y = f(x) = 10^x.$$

Our claim is that this function f grows faster than $f_n(x) = x^n$ for any positive integer n. For example, we can prove that 10^x grows faster than $f_{100}(x) = x^{100}$. For large values of x like $x = 10^3$, note that $f_{100}(x) = (10^3)^{100} = 10^{300}$ whereas $f(10^3) = 10^{10^3} = 10^{1000}$. Clearly $f(x)$ is much greater than $f_{100}(x)$. It is not difficult to prove that for all $x > 10^3$, $f(x) > f_{100}(x)$. But we will not attempt to give a proof of this here. Similarly, by choosing large values of x, one can verify that $f(x)$ grows faster than $f_n(x)$ for any positive integer n.

Definition 3 The exponential function with positive base $b > 1$ is the function

$$y = f(x) = b^x$$

The graph of $y = 10^x$ is given in the Fig 5.9.

It is advised that the reader plots this graph for particular values of b like 2, 3 and 4. Following are some of the salient features of the exponential functions:

(1) Domain of the exponential function is **R**, the set of all real numbers.

(2) Range of the exponential function is the set of all positive real numbers.

(3) The point (0, 1) is always on the graph of the exponential function (this is a restatement of the fact that $b^0 = 1$ for any real $b > 1$).

(4) Exponential function is ever increasing; i.e., as we move from left to right, the graph rises above.

(5) For very large negative values of x, the exponential function is very close to 0. In other words, in the second quadrant, the graph approaches x-axis (but never meets it).

Exponential function with base 10 is called the *common exponential function*. In the Appendix A.1.4 of Class XI, it was observed that the sum of the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + ...$$

is a number between 2 and 3 and is denoted by e. Using this e as the base we obtain an extremely important exponential function $y = e^x$.

This is called *natural exponential function*.

It would be interesting to know if the inverse of the exponential function exists and has *nice* interpretation. This search motivates the following definition.

Definition 4 Let $b > 1$ be a real number. Then we say logarithm of a to base b is x if $b^x = a$.

Logarithm of a to base b is denoted by $\log_b a$. Thus $\log_b a = x$ if $b^x = a$. Let us work with a few explicit examples to get a feel for this. We know $2^3 = 8$. In terms of logarithms, we may rewrite this as $\log_2 8 = 3$. Similarly, $10^4 = 10000$ is equivalent to saying $\log_{10} 10000 = 4$. Also, $625 = 5^4 = 25^2$ is equivalent to saying $\log_5 625 = 4$ or $\log_{25} 625 = 2$.

On a slightly more mature note, fixing a base $b > 1$, we may look at logarithm as a function from positive real numbers to all real numbers. This function, called the *logarithmic function*, is defined by

$$\log_b : \mathbf{R}^+ \to \mathbf{R}$$
$$x \to \log_b x = y \quad \text{if } b^y = x$$

As before if the base $b = 10$, we say it is *common logarithms* and if $b = e$, then we say it is *natural logarithms*. Often natural logarithm is denoted by *ln*. *In this chapter, log x denotes the logarithm function to base e*, i.e., *ln x* will be written as simply log *x*. The Fig 5.10 gives the plots of logarithm function to base 2, *e* and 10.

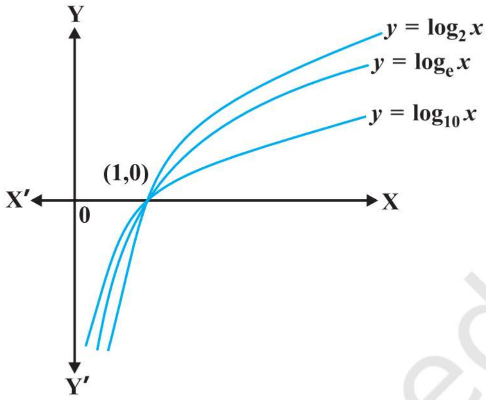

Fig 5.10

Some of the important observations about the logarithm function to any base $b > 1$ are listed below:

(1) We cannot make a meaningful definition of logarithm of non-positive numbers and hence the domain of log function is $\mathbf{R}^+$.

(2) The range of log function is the set of all real numbers.

(3) The point (1, 0) is always on the graph of the log function.

(4) The log function is ever increasing, i.e., as we move from left to right the graph rises above.

(5) For *x* very near to zero, the value of log *x* can be made lesser than any given real number. In other words in the fourth quadrant the graph approaches *y*-axis (but never meets it).

(6) Fig 5.11 gives the plot of $y = e^x$ and $y = ln\ x$. It is of interest to observe that the two curves are the mirror images of each other reflected in the line $y = x$.

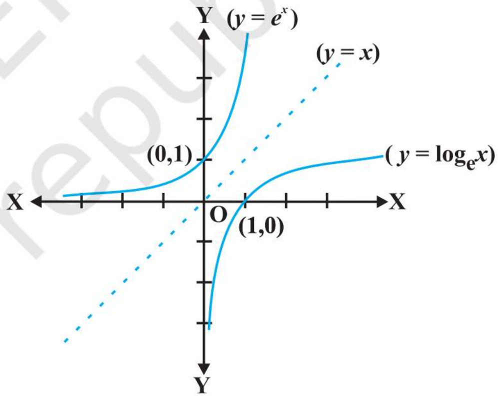

Fig 5.11

Two properties of 'log' functions are proved below:

(1) There is a standard change of base rule to obtain $\log_a p$ in terms of $\log_b p$. Let $\log_a p = \alpha$, $\log_b p = \beta$ and $\log_b a = \gamma$. This means $a^\alpha = p$, $b^\beta = p$ and $b^\gamma = a$.

Substituting the third equation in the first one, we have

$$(b^\gamma)^\alpha = b^{\gamma\alpha} = p$$

Using this in the second equation, we get

$$b^\beta = p = b^{\gamma\alpha}$$

which implies $\beta = \alpha\gamma$ or $\alpha = \frac{\beta}{\gamma}$. But then

$$\log_a p = \frac{\log_b p}{\log_b a}$$

(2) Another interesting property of the log function is its effect on products. Let $\log_b pq = \alpha$. Then $b^\alpha = pq$. If $\log_b p = \beta$ and $\log_b q = \gamma$, then $b^\beta = p$ and $b^\gamma = q$. But then $b^\alpha = pq = b^\beta b^\gamma = b^{\beta+\gamma}$

which implies $\alpha = \beta + \gamma$, i.e.,

$$\log_b pq = \log_b p + \log_b q$$

A particularly interesting and important consequence of this is when $p = q$. In this case the above may be rewritten as

$$\log_b p^2 = \log_b p + \log_b p = 2 \log p$$

An easy generalisation of this (left as an exercise!) is

$$\log_b p^n = n \log p$$

for any positive integer n. In fact this is true for any real number n, but we will not attempt to prove this. On the similar lines the reader is invited to verify

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

Example 28 Is it true that $x = e^{\log x}$ for all real x?

Solution First, observe that the domain of log function is set of all positive real numbers. So the above equation is not true for non-positive real numbers. Now, let $y = e^{\log x}$. If $y > 0$, we may take logarithm which gives us $\log y = \log (e^{\log x}) = \log x \,.\, \log e = \log x$. Thus $y = x$. Hence $x = e^{\log x}$ is true only for positive values of x.

One of the striking properties of the natural exponential function in differential calculus is that it doesn't change during the process of differentiation. This is captured in the following theorem whose proof we skip.

Theorem 5*

(1) The derivative of e^x w.r.t., x is e^x; i.e., $\frac{d}{dx}(e^x) = e^x$.

(2) The derivative of $\log x$ w.r.t., x is $\frac{1}{x}$; i.e., $\frac{d}{dx}(\log x) = \frac{1}{x}$.

* *Please see supplementary material on Page 286.*

Example 29 Differentiate the following w.r.t. x:

(i) e^{-x} (ii) $\sin(\log x), x > 0$ (iii) $\cos^{-1}(e^x)$ (iv) $e^{\cos x}$

Solution

(i) Let $y = e^{-x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{-x} \cdot \frac{d}{dx}(-x) = -e^{-x}$$

(ii) Let $y = \sin(\log x)$. Using chain rule, we have

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{d}{dx}(\log x) = \frac{\cos(\log x)}{x}$$

(iii) Let $y = \cos^{-1}(e^x)$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(e^x)^2}} \cdot \frac{d}{dx}(e^x) = \frac{-e^x}{\sqrt{1-e^{2x}}}$$

(iv) Let $y = e^{\cos x}$. Using chain rule, we have

$$\frac{dy}{dx} = e^{\cos x} \cdot (-\sin x) = -(\sin x)\, e^{\cos x}$$

EXERCISE 5.4

Differentiate the following w.r.t. x:

1. $\dfrac{e^x}{\sin x}$ **2.** $e^{\sin^{-1} x}$ **3.** e^{x^3}

4. $\sin(\tan^{-1} e^{-x})$ **5.** $\log(\cos e^x)$ **6.** $e^x + e^{x^2} + \ldots + e^{x^5}$

7. $\sqrt{e^{\sqrt{x}}},\ x > 0$ **8.** $\log(\log x), x > 1$ **9.** $\dfrac{\cos x}{\log x},\ x > 0$

10. $\cos(\log x + e^x), x > 0$

5.5. Logarithmic Differentiation

In this section, we will learn to differentiate certain special class of functions given in the form

$$y = f(x) = [u(x)]^{v(x)}$$

By taking logarithm (to base e) the above may be rewritten as

$$\log y = v(x) \log [u(x)]$$