Write the first five terms of each of the sequences in Exercises 11 to 13 and obtain the corresponding series:

11. $a_1 = 3$, $a_n = 3a_{n-1} + 2$ for all $n > 1$

12. $a_1 = -1$, $a_n = \dfrac{a_{n-1}}{n}$, $n \geq 2$

13. $a_1 = a_2 = 2$, $a_n = a_{n-1} - 1$, $n > 2$

14. The Fibonacci sequence is defined by

$$1 = a_1 = a_2 \text{ and } a_n = a_{n-1} + a_{n-2},\ n > 2.$$

Find $\dfrac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, 4, 5$

9.4 Arithmetic Progression (A.P.)

Let us recall some formulae and properties studied earlier.

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called *arithmetic sequence or arithmetic progression* if $a_{n+1} = a_n + d$, $n \in \mathbf{N}$, where a_1 is called the *first term* and the constant term d is called the *common difference* of the A.P.

Let us consider an A.P. (in its standard form) with first term a and common difference d, i.e., $a, a + d, a + 2d, \ldots$

Then the n^{th} term (*general term*) of the A.P. is $a_n = a + (n - 1)\, d$.

We can verify the following simple properties of an A.P. :

(i) If a constant is added to each term of an A.P., the resulting sequence is also an A.P.

(ii) If a constant is subtracted from each term of an A.P., the resulting sequence is also an A.P.

(iii) If each term of an A.P. is multiplied by a constant, then the resulting sequence is also an A.P.

(iv) If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

Here, we shall use the following notations for an arithmetic progression:

a = the first term, l = the last term, d = common difference,

n = the number of terms.

S_n = the sum to n terms of A.P.

Let $a, a + d, a + 2d, \ldots, a + (n - 1)\, d$ be an A.P. Then

$$l = a + (n - 1)\, d$$

$$S_n = \frac{n}{2}[2a+(n-1)d]$$

We can also write, $S_n = \frac{n}{2}[a+l]$

Let us consider some examples.

Example 4 In an A.P. if m^{th} term is n and the n^{th} term is m, where $m \neq n$, find the pth term.

Solution We have $a_m = a + (m - 1)\, d = n,$ ... (1)

and $a_n = a + (n - 1)\, d = m$ ... (2)

Solving (1) and (2), we get

$(m - n)\, d = n - m$, or $d = -1,$ ... (3)

and $a = n + m - 1$ ... (4)

Therefore $a_p = a + (p - 1)d$

$= n + m - 1 + (p - 1)(-1) = n + m - p$

Hence, the p^{th} term is $n + m - p$.

Example 5 If the sum of n terms of an A.P. is $nP + \frac{1}{2}n(n-1)Q$, where P and Q are constants, find the common difference.

Solution Let $a_1, a_2, \dots a_n$ be the given A.P. Then

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n = nP + \frac{1}{2}n\,(n-1)\,Q$$

Therefore $S_1 = a_1 = P$, $S_2 = a_1 + a_2 = 2P + Q$

So that $a_2 = S_2 - S_1 = P + Q$

Hence, the common difference is given by $d = a_2 - a_1 = (P + Q) - P = Q$.

Example 6 The sum of n terms of two arithmetic progressions are in the ratio $(3n + 8) : (7n + 15)$. Find the ratio of their 12^{th} terms.

Solution Let a_1, a_2 and d_1, d_2 be the first terms and common difference of the first and second arithmetic progression, respectively. According to the given condition, we have

$$\frac{\text{Sum to } n \text{ terms of first A.P.}}{\text{Sum to } n \text{ terms of second A.P.}} = \frac{3n+8}{7n+15}$$

or $$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]}=\frac{3n+8}{7n+15}$$

or $$\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{3n+8}{7n+15} \qquad \text{... (1)}$$

Now $$\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P}}=\frac{a_1+11d_1}{a_2+11d_2}$$

$$\frac{2a_1+22d_1}{2a_2+22d_2}=\frac{3\times 23+8}{7\times 23+15} \qquad \text{[By putting } n = 23 \text{ in (1)]}$$

Therefore $$\frac{a_1+11d_1}{a_2+11d_2}=\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P.}}=\frac{7}{16}$$

Hence, the required ratio is 7 : 16.

Example 7 The income of a person is Rs. 3,00,000, in the first year and he receives an increase of Rs.10,000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.

Solution Here, we have an A.P. with a = 3,00,000, d = 10,000, and n = 20.
Using the sum formula, we get,

$$S_{20}=\frac{20}{2}[600000+19\times 10000] = 10\,(790000) = 79,00,000.$$

Hence, the person received Rs. 79,00,000 as the total amount at the end of 20 years.

9.4.1 ***Arithmetic mean*** Given two numbers a and b. We can insert a number A between them so that a, A, b is an A.P. Such a number A is called the *arithmetic mean* (A.M.) of the numbers a and b. Note that, in this case, we have

$$A - a = b - A, \quad \text{i.e., } A = \frac{a+b}{2}$$

We may also interpret the A.M. between two numbers a and b as their average $\frac{a+b}{2}$. For example, the A.M. of two numbers 4 and 16 is 10. We have, thus constructed an A.P. 4, 10, 16 by inserting a number 10 between 4 and 16. The natural