Find n, so that $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ $(a \neq b)$ be HM between a and b.

SOLUTION :

$$\because \quad \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{2ab}{a+b}$$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\dfrac{a}{b}\right)^{n+1} + 1\right]}{b^n\left[\left(\dfrac{a}{b}\right)^{n} + 1\right]} = \frac{b^2\left[2\left(\dfrac{a}{b}\right)\right]}{b\left(\dfrac{a}{b} + 1\right)}$$

$$\Rightarrow \quad \frac{\left(\dfrac{a}{b}\right)^{n+1} + 1}{\left(\dfrac{a}{b}\right)^{n} + 1} = \frac{2\left(\dfrac{a}{b}\right)}{\left(\dfrac{a}{b}\right) + 1}$$

Let $\quad \dfrac{a}{b} = \lambda$

Then, $\quad \dfrac{\lambda^{n+1} + 1}{\lambda^n + 1} = \dfrac{2\lambda}{\lambda + 1}$

$\Rightarrow \quad (\lambda + 1)(\lambda^{n+1} + 1) = 2\lambda(\lambda^n + 1)$

$\Rightarrow \quad \lambda^{n+2} + \lambda + \lambda^{n+1} + 1 = 2\lambda^{n+1} + 2\lambda$

$\Rightarrow \quad \lambda^{n+2} - \lambda^{n+1} - \lambda + 1 = 0$

$\Rightarrow \quad \lambda^{n+1}(\lambda - 1) - 1(\lambda - 1) = 0$

$\Rightarrow \quad (\lambda - 1)(\lambda^{n+1} - 1) = 0$

$\Rightarrow \quad \lambda - 1 \neq 0 \qquad\qquad [\because a \neq b]$

$\therefore \quad \lambda^{n+1} - 1 = 0$