or $$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]}=\frac{3n+8}{7n+15}$$

or $$\frac{2a_1+(n-1)d_1}{2a_2+(n-1)d_2}=\frac{3n+8}{7n+15} \quad \text{... (1)}$$

Now $$\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P}}=\frac{a_1+11d_1}{a_2+11d_2}$$

$$\frac{2a_1+22d_1}{2a_2+22d_2}=\frac{3\times 23+8}{7\times 23+15} \quad \text{[By putting } n = 23 \text{ in (1)]}$$

Therefore $$\frac{a_1+11d_1}{a_2+11d_2}=\frac{12^{\text{th}}\text{ term of first A.P.}}{12^{\text{th}}\text{ term of second A.P.}}=\frac{7}{16}$$

Hence, the required ratio is 7 : 16.

Example 7 The income of a person is Rs. 3,00,000, in the first year and he receives an increase of Rs.10,000 to his income per year for the next 19 years. Find the total amount, he received in 20 years.

Solution Here, we have an A.P. with $a = 3{,}00{,}000$, $d = 10{,}000$, and $n = 20$.
Using the sum formula, we get,

$$S_{20}=\frac{20}{2}[600000+19\times 10000]=10\,(790000)=79{,}00{,}000.$$

Hence, the person received Rs. 79,00,000 as the total amount at the end of 20 years.

9.4.1 ***Arithmetic mean*** Given two numbers a and b. We can insert a number A between them so that a, A, b is an A.P. Such a number A is called the *arithmetic mean* (A.M.) of the numbers a and b. Note that, in this case, we have

$$A - a = b - A, \quad \text{i.e., } A = \frac{a+b}{2}$$

We may also interpret the A.M. between two numbers a and b as their average $\frac{a+b}{2}$. For example, the A.M. of two numbers 4 and 16 is 10. We have, thus constructed an A.P. 4, 10, 16 by inserting a number 10 between 4 and 16. The natural question now arises : Can we insert two or more numbers between given two numbers so that the resulting sequence comes out to be an A.P. ? Observe that two numbers 8 and 12 can be inserted between 4 and 16 so that the resulting sequence 4, 8, 12, 16 becomes an A.P.

More generally, given any two numbers a and b, we can insert as many numbers as we like between them such that the resulting sequence is an A.P.

Let $A_1, A_2, A_3, \ldots, A_n$ be n numbers between a and b such that $a, A_1, A_2, A_3, \ldots, A_n, b$ is an A.P.

Here, b is the $(n + 2)^{\text{th}}$ term, i.e., $b = a + [(n + 2) - 1]d = a + (n + 1) d.$

This gives $$d = \frac{b-a}{n+1}.$$

Thus, n numbers between a and b are as follows:

$$A_1 = a + d = a + \frac{b-a}{n+1}$$

$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$$

$$A_3 = a + 3d = a + \frac{3(b-a)}{n+1}$$

....

....

$$A_n = a + nd = a + \frac{n(b-a)}{n+1}.$$

Example 8 Insert 6 numbers between 3 and 24 such that the resulting sequence is an A.P.

Solution Let A_1, A_2, A_3, A_4, A_5 and A_6 be six numbers between 3 and 24 such that

$3, A_1, A_2, A_3, A_4, A_5, A_6, 24$ are in A.P. Here, $a = 3$, $b = 24$, $n = 8$.

Therefore, $24 = 3 + (8 - 1) d$, so that $d = 3$.

Thus $A_1 = a + d = 3 + 3 = 6$; $A_2 = a + 2d = 3 + 2 \times 3 = 9$;

$A_3 = a + 3d = 3 + 3 \times 3 = 12$; $A_4 = a + 4d = 3 + 4 \times 3 = 15$;

$A_5 = a + 5d = 3 + 5 \times 3 = 18$; $A_6 = a + 6d = 3 + 6 \times 3 = 21$.

Hence, six numbers between 3 and 24 are 6, 9, 12, 15, 18 and 21.

$$= \frac{7}{9}[(10+10^2+10^3+\ldots n \text{ terms})-(1+1+1+\ldots n \text{ terms})]$$

$$= \frac{7}{9}\left[\frac{10(10^n-1)}{10-1}-n\right] = \frac{7}{9}\left[\frac{10\,(10^n-1)}{9}-n\right].$$

Example 16 A person has 2 parents, 4 grandparents, 8 great grandparents, and so on. Find the number of his ancestors during the ten generations preceding his own.

Solution Here $a = 2$, $r = 2$ and $n = 10$

Using the sum formula $\quad S_n = \dfrac{a\,(r^n-1)}{r-1}$

We have $\quad S_{10} = 2(2^{10} - 1) = 2046$

Hence, the number of ancestors preceding the person is 2046.

9.5.3 ***Geometric Mean (G.M.)*** The geometric mean of two positive numbers a and b is the number $\sqrt{ab}$. Therefore, the geometric mean of 2 and 8 is 4. We observe that the three numbers 2,4,8 are consecutive terms of a G.P. This leads to a generalisation of the concept of geometric means of two numbers.

Given any two positive numbers a and b, we can insert as many numbers as we like between them to make the resulting sequence in a G.P.

Let $G_1, G_2, \ldots, G_n$ be n numbers between positive numbers a and b such that $a, G_1, G_2, G_3, \ldots, G_n, b$ is a G.P. Thus, b being the $(n + 2)^{th}$ term,we have

$$b = ar^{n+1}, \quad \text{or} \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence $\quad G_1 = ar = a\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}, \quad G_2 = ar^2 = a\left(\dfrac{b}{a}\right)^{\frac{2}{n+1}}, \quad G_3 = ar^3 = a\left(\dfrac{b}{a}\right)^{\frac{3}{n+1}},$

$$G_n = ar^n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

Example17 Insert three numbers between 1 and 256 so that the resulting sequence is a G.P.

Solution Let G_1, G_2, G_3 be three numbers between 1 and 256 such that $1, G_1, G_2, G_3, 256$ is a G.P.

Therefore $256 = r^4$ giving $r = \pm 4$ (Taking real roots only)

For $r = 4$, we have $G_1 = ar = 4$, $G_2 = ar^2 = 16$, $G_3 = ar^3 = 64$

Similarly, for $r = -4$, numbers are $-4, 16$ and -64.

Hence, we can insert 4, 16, 64 between 1 and 256 so that the resulting sequences are in G.P.

9.6 Relationship Between A.M. and G.M.

Let A and G be A.M. and G.M. of two given positive real numbers a and b, respectively. Then

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

Thus, we have

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0 \qquad \text{... (1)}$$

From (1), we obtain the relationship $A \geq G$.

Example 18 If A.M. and G.M. of two positive numbers a and b are 10 and 8, respectively, find the numbers.

Solution Given that $$\text{A.M.} = \frac{a+b}{2} = 10 \qquad \text{... (1)}$$

and $$\text{G.M.} = \sqrt{ab} = 8 \qquad \text{... (2)}$$

From (1) and (2), we get

$$a + b = 20 \qquad \text{... (3)}$$

$$ab = 64 \qquad \text{... (4)}$$

Putting the value of a and b from (3), (4) in the identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a - b)^2 = 400 - 256 = 144$$

or $$a - b = \pm 12$$
... (5)

Solving (3) and (5), we obtain

$$a = 4, b = 16 \text{ or } a = 16, b = 4$$

Thus, the numbers a and b are 4, 16 or 16, 4 respectively.