



CONTINUITY AND DIFFERENTIABILITY

❖ *The whole of science is nothing more than a refinement of everyday thinking.* — ALBERT EINSTEIN ❖

5.1 Introduction

This chapter is essentially a continuation of our study of differentiation of functions in Class XI. We had learnt to differentiate certain functions like polynomial functions and trigonometric functions. In this chapter, we introduce the very important concepts of continuity, differentiability and relations between them. We will also learn differentiation of inverse trigonometric functions. Further, we introduce a new class of functions called exponential and logarithmic functions. These functions lead to powerful techniques of differentiation. We illustrate certain geometrically obvious conditions through differential calculus. In the process, we will learn some fundamental theorems in this area.



Sir Issac Newton
(1642-1727)

5.2 Continuity

We start the section with two informal examples to get a feel of continuity. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \leq 0 \\ 2, & \text{if } x > 0 \end{cases}$$

This function is of course defined at every point of the real line. Graph of this function is given in the Fig 5.1. One can deduce from the graph that the value of the function at *nearby* points on *x*-axis remain *close* to each other except at $x = 0$. At the points near and to the left of 0, i.e., at points like $-0.1, -0.01, -0.001$, the value of the function is 1. At the points near and to the right of 0, i.e., at points like $0.1, 0.01,$

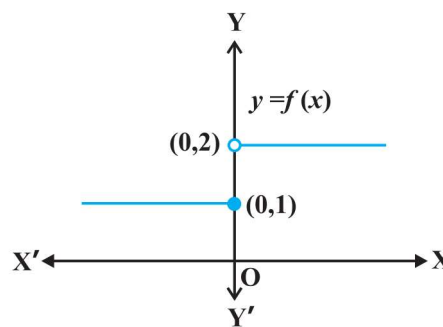


Fig 5.1

0.001, the value of the function is 2. Using the language of left and right hand limits, we may say that the left (respectively right) hand limit of f at 0 is 1 (respectively 2). In particular the left and right hand limits do not coincide. We also observe that the value of the function at $x = 0$ coincides with the left hand limit. Note that when we try to draw the graph, we cannot draw it in one stroke, i.e., without lifting pen from the plane of the paper, we can not draw the graph of this function. In fact, we need to lift the pen when we come to 0 from left. This is one instance of function being not continuous at $x = 0$.

Now, consider the function defined as

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$

This function is also defined at every point. Left and the right hand limits at $x = 0$ are both equal to 1. But the value of the function at $x = 0$ equals 2 which does not coincide with the common value of the left and right hand limits. Again, we note that we cannot draw the graph of the function without lifting the pen. This is yet another instance of a function being not continuous at $x = 0$.

Naively, we may say that a function is continuous at a fixed point if we can draw the graph of the function *around* that point without lifting the pen from the plane of the paper.

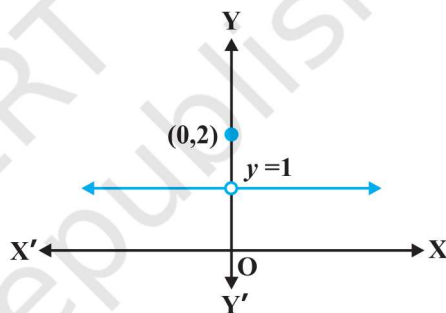


Fig 5.2

Mathematically, it may be phrased precisely as follows:

Definition 1 Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f . Then f is continuous at c if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

More elaborately, if the left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other, then f is said to be continuous at $x = c$. Recall that if the right hand and left hand limits at $x = c$ coincide, then we say that the common value is the limit of the function at $x = c$. Hence we may also rephrase the definition of continuity as follows: *a function is continuous at $x = c$ if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$* . If f is not continuous at c , we say f is *discontinuous* at c and c is called a *point of discontinuity* of f .

Example 1 Check the continuity of the function f given by $f(x) = 2x + 3$ at $x = 1$.

Solution First note that the function is defined at the given point $x = 1$ and its value is 5. Then find the limit of the function at $x = 1$. Clearly

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (2x + 3) = 2(1) + 3 = 5$$

Thus $\lim_{x \rightarrow 1} f(x) = 5 = f(1)$

Hence, f is continuous at $x = 1$.

Example 2 Examine whether the function f given by $f(x) = x^2$ is continuous at $x = 0$.

Solution First note that the function is defined at the given point $x = 0$ and its value is 0. Then find the limit of the function at $x = 0$. Clearly

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^2 = 0^2 = 0$$

Thus $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$

Hence, f is continuous at $x = 0$.

Example 3 Discuss the continuity of the function f given by $f(x) = |x|$ at $x = 0$.

Solution By definition

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly the function is defined at 0 and $f(0) = 0$. Left hand limit of f at 0 is

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

Similarly, the right hand limit of f at 0 is

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

Thus, the left hand limit, right hand limit and the value of the function coincide at $x = 0$. Hence, f is continuous at $x = 0$.

Example 4 Show that the function f given by

$$f(x) = \begin{cases} x^3 + 3, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

is not continuous at $x = 0$.

Solution The function is defined at $x = 0$ and its value at $x = 0$ is 1. When $x \neq 0$, the function is given by a polynomial. Hence,

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (x^3 + 3) = 0^3 + 3 = 3$$

Since the limit of f at $x = 0$ does not coincide with $f(0)$, the function is not continuous at $x = 0$. It may be noted that $x = 0$ is the only point of discontinuity for this function.

Example 5 Check the points where the constant function $f(x) = k$ is continuous.

Solution The function is defined at all real numbers and by definition, its value at any real number equals k . Let c be any real number. Then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} k = k$$

Since $f(c) = k = \lim_{x \rightarrow c} f(x)$ for any real number c , the function f is continuous at every real number.

Example 6 Prove that the identity function on real numbers given by $f(x) = x$ is continuous at every real number.

Solution The function is clearly defined at every point and $f(c) = c$ for every real number c . Also,

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c$$

Thus, $\lim_{x \rightarrow c} f(x) = c = f(c)$ and hence the function is continuous at every real number.

Having defined continuity of a function at a given point, now we make a natural extension of this definition to discuss continuity of a function.

Definition 2 A real function f is said to be continuous if it is continuous at every point in the domain of f .

This definition requires a bit of elaboration. Suppose f is a function defined on a closed interval $[a, b]$, then for f to be continuous, it needs to be continuous at every point in $[a, b]$ including the end points a and b . Continuity of f at a means

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$

and continuity of f at b means

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$

Observe that $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow b^+} f(x)$ do not make sense. As a consequence of this definition, if f is defined only at one point, it is continuous there, i.e., if the domain of f is a singleton, f is a continuous function.

Example 7 Is the function defined by $f(x) = |x|$, a continuous function?

Solution We may rewrite f as

$$f(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

By Example 3, we know that f is continuous at $x = 0$.

Let c be a real number such that $c < 0$. Then $f(c) = -c$. Also

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-x) = -c \quad (\text{Why?})$$

Since $\lim_{x \rightarrow c} f(x) = f(c)$, f is continuous at all negative real numbers.

Now, let c be a real number such that $c > 0$. Then $f(c) = c$. Also

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c \quad (\text{Why?})$$

Since $\lim_{x \rightarrow c} f(x) = f(c)$, f is continuous at all positive real numbers. Hence, f is continuous at all points.

Example 8 Discuss the continuity of the function f given by $f(x) = x^3 + x^2 - 1$.

Solution Clearly f is defined at every real number c and its value at c is $c^3 + c^2 - 1$. We also know that

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x^3 + x^2 - 1) = c^3 + c^2 - 1$$

Thus $\lim_{x \rightarrow c} f(x) = f(c)$, and hence f is continuous at every real number. This means f is a continuous function.

Example 9 Discuss the continuity of the function f defined by $f(x) = \frac{1}{x}$, $x \neq 0$.

Solution Fix any non zero real number c , we have

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}$$

Also, since for $c \neq 0$, $f(c) = \frac{1}{c}$, we have $\lim_{x \rightarrow c} f(x) = f(c)$ and hence, f is continuous at every point in the domain of f . Thus f is a continuous function.

We take this opportunity to explain the concept of *infinity*. This we do by analysing the function $f(x) = \frac{1}{x}$ near $x = 0$. To carry out this analysis we follow the usual trick of finding the value of the function at real numbers *close* to 0. Essentially we are trying to find the right hand limit of f at 0. We tabulate this in the following (Table 5.1).

Table 5.1

x	1	0.3	0.2	$0.1 = 10^{-1}$	$0.01 = 10^{-2}$	$0.001 = 10^{-3}$	10^{-n}
$f(x)$	1	3.333...	5	10	$100 = 10^2$	$1000 = 10^3$	10^n

We observe that as x gets closer to 0 from the right, the value of $f(x)$ shoots up higher. This may be rephrased as: the value of $f(x)$ may be made larger than any given number by choosing a positive real number *very close* to 0. In symbols, we write

$$\lim_{x \rightarrow 0^+} f(x) = +\infty$$

(to be read as: the right hand limit of $f(x)$ at 0 is plus infinity). We wish to emphasise that $+\infty$ is NOT a real number and hence the right hand limit of f at 0 does not exist (as a real number).

Similarly, the left hand limit of f at 0 may be found. The following table is self explanatory.

Table 5.2

x	-1	-0.3	-0.2	-10^{-1}	-10^{-2}	-10^{-3}	-10^{-n}
$f(x)$	-1	-3.333...	-5	-10	-10^2	-10^3	-10^n

From the Table 5.2, we deduce that the value of $f(x)$ may be made smaller than any given number by choosing a negative real number *very close* to 0. In symbols, we write

$$\lim_{x \rightarrow 0^-} f(x) = -\infty$$

(to be read as: the left hand limit of $f(x)$ at 0 is minus infinity). Again, we wish to emphasise that $-\infty$ is NOT a real number and hence the left hand limit of f at 0 does not exist (as a real number). The graph of the reciprocal function given in Fig 5.3 is a geometric representation of the above mentioned facts.

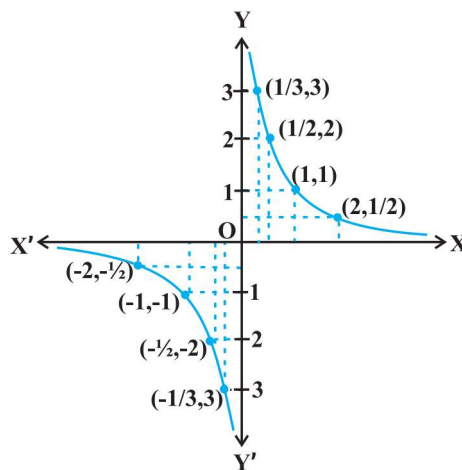


Fig 5.3

Example 10 Discuss the continuity of the function f defined by

$$f(x) = \begin{cases} x + 2, & \text{if } x \leq 1 \\ x - 2, & \text{if } x > 1 \end{cases}$$

Solution The function f is defined at all points of the real line.

Case 1 If $c < 1$, then $f(c) = c + 2$. Therefore, $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 2) = c + 2$

Thus, f is continuous at all real numbers less than 1.

Case 2 If $c > 1$, then $f(c) = c - 2$. Therefore,

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x - 2) = c - 2 = f(c)$$

Thus, f is continuous at all points $x > 1$.

Case 3 If $c = 1$, then the left hand limit of f at $x = 1$ is

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 2) = 1 + 2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x - 2) = 1 - 2 = -1$$

Since the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f . The graph of the function is given in Fig 5.4.

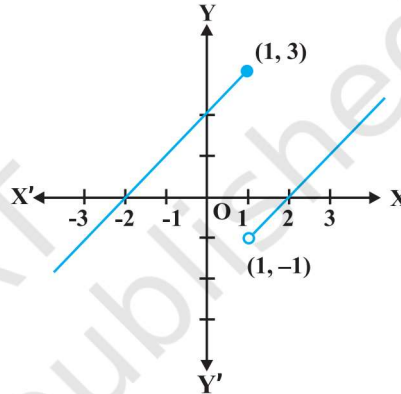


Fig 5.4

Example 11 Find all the points of discontinuity of the function f defined by

$$f(x) = \begin{cases} x + 2, & \text{if } x < 1 \\ 0, & \text{if } x = 1 \\ x - 2, & \text{if } x > 1 \end{cases}$$

Solution As in the previous example we find that f is continuous at all real numbers $x \neq 1$. The left hand limit of f at $x = 1$ is

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x + 2) = 1 + 2 = 3$$

The right hand limit of f at $x = 1$ is

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x - 2) = 1 - 2 = -1$$

Since, the left and right hand limits of f at $x = 1$ do not coincide, f is not continuous at $x = 1$. Hence $x = 1$ is the only point of discontinuity of f . The graph of the function is given in the Fig 5.5.

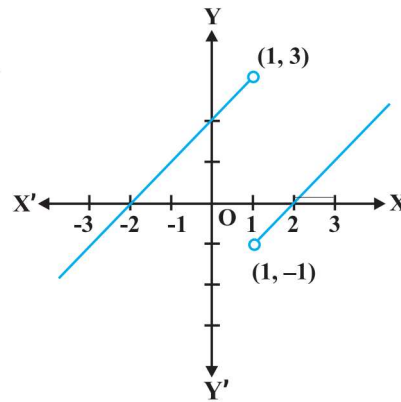


Fig 5.5

Example 12 Discuss the continuity of the function defined by

$$f(x) = \begin{cases} x+2, & \text{if } x < 0 \\ -x+2, & \text{if } x > 0 \end{cases}$$

Solution Observe that the function is defined at all real numbers except at 0. Domain of definition of this function is

$$D_1 \cup D_2 \text{ where } D_1 = \{x \in \mathbf{R} : x < 0\} \text{ and} \\ D_2 = \{x \in \mathbf{R} : x > 0\}$$

Case 1 If $c \in D_1$, then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (x + 2)$
 $= c + 2 = f(c)$ and hence f is continuous in D_1 .

Case 2 If $c \in D_2$, then $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} (-x + 2)$
 $= -c + 2 = f(c)$ and hence f is continuous in D_2 .

Since f is continuous at all points in the domain of f , we deduce that f is continuous. Graph of this function is given in the Fig 5.6. Note that to graph this function we need to lift the pen from the plane of the paper, but we need to do that only for those points where the function is not defined.

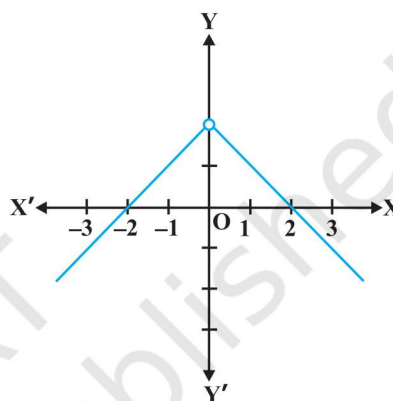


Fig 5.6

Example 13 Discuss the continuity of the function f given by

$$f(x) = \begin{cases} x, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$$

Solution Clearly the function is defined at every real number. Graph of the function is given in Fig 5.7. By inspection, it seems prudent to partition the domain of definition of f into three disjoint subsets of the real line.

Let $D_1 = \{x \in \mathbf{R} : x < 0\}$, $D_2 = \{0\}$ and
 $D_3 = \{x \in \mathbf{R} : x > 0\}$

Case 1 At any point in D_1 , we have $f(x) = x^2$ and it is easy to see that it is continuous there (see Example 2).

Case 2 At any point in D_3 , we have $f(x) = x$ and it is easy to see that it is continuous there (see Example 6).

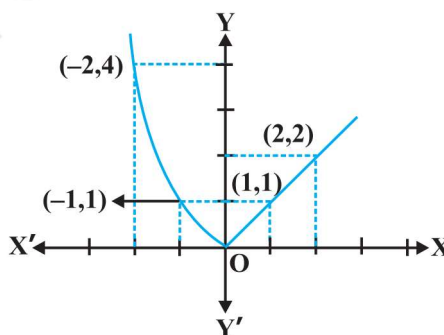


Fig 5.7

Case 3 Now we analyse the function at $x = 0$. The value of the function at 0 is $f(0) = 0$. The left hand limit of f at 0 is

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0^2 = 0$$

The right hand limit of f at 0 is

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

Thus $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ and hence f is continuous at 0. This means that f is continuous at every point in its domain and hence, f is a continuous function.

Example 14 Show that every polynomial function is continuous.

Solution Recall that a function p is a polynomial function if it is defined by $p(x) = a_0 + a_1 x + \dots + a_n x^n$ for some natural number n , $a_n \neq 0$ and $a_i \in \mathbf{R}$. Clearly this function is defined for every real number. For a fixed real number c , we have

$$\lim_{x \rightarrow c} p(x) = p(c)$$

By definition, p is continuous at c . Since c is any real number, p is continuous at every real number and hence p is a continuous function.

Example 15 Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x .

Solution First observe that f is defined for all real numbers. Graph of the function is given in Fig 5.8. From the graph it looks like that f is discontinuous at every integral point. Below we explore, if this is true.

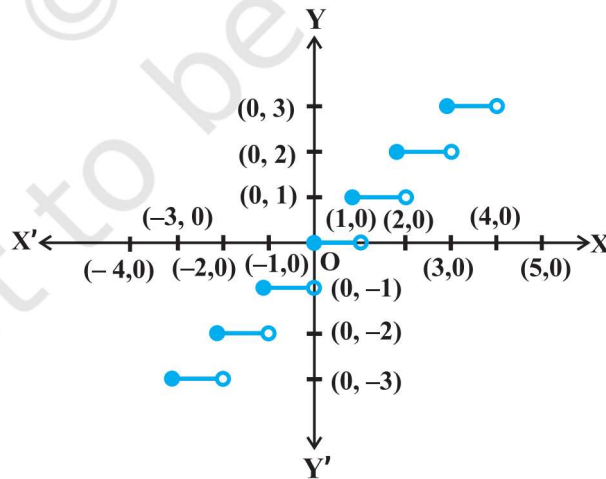


Fig 5.8

Case 1 Let c be a real number which is not equal to any integer. It is evident from the graph that for all real numbers *close* to c the value of the function is equal to $[c]$; i.e.,

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} [x] = [c]. \text{ Also } f(c) = [c] \text{ and hence the function is continuous at all real}$$

numbers not equal to integers.

Case 2 Let c be an integer. Then we can find a sufficiently small real number $r > 0$ such that $[c - r] = c - 1$ whereas $[c + r] = c$.

This, in terms of limits mean that

$$\lim_{x \rightarrow c^-} f(x) = c - 1, \quad \lim_{x \rightarrow c^+} f(x) = c$$

Since these limits cannot be equal to each other for any c , the function is discontinuous at every integral point.

5.2.1 Algebra of continuous functions

In the previous class, after having understood the concept of limits, we learnt some algebra of limits. Analogously, now we will study some algebra of continuous functions. Since continuity of a function at a point is entirely dictated by the limit of the function at that point, it is reasonable to expect results analogous to the case of limits.

Theorem 1 Suppose f and g be two real functions continuous at a real number c . Then

- (1) $f + g$ is continuous at $x = c$.
- (2) $f - g$ is continuous at $x = c$.
- (3) $f \cdot g$ is continuous at $x = c$.
- (4) $\left(\frac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).

Proof We are investigating continuity of $(f + g)$ at $x = c$. Clearly it is defined at $x = c$. We have

$$\begin{aligned} \lim_{x \rightarrow c} (f + g)(x) &= \lim_{x \rightarrow c} [f(x) + g(x)] && \text{(by definition of } f + g) \\ &= \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x) && \text{(by the theorem on limits)} \\ &= f(c) + g(c) && \text{(as } f \text{ and } g \text{ are continuous)} \\ &= (f + g)(c) && \text{(by definition of } f + g) \end{aligned}$$

Hence, $f + g$ is continuous at $x = c$.

Proofs for the remaining parts are similar and left as an exercise to the reader.

Remarks

- (i) As a special case of (3) above, if f is a constant function, i.e., $f(x) = \lambda$ for some real number λ , then the function $(\lambda \cdot g)$ defined by $(\lambda \cdot g)(x) = \lambda \cdot g(x)$ is also continuous. In particular if $\lambda = -1$, the continuity of f implies continuity of $-f$.
- (ii) As a special case of (4) above, if f is the constant function $f(x) = \lambda$, then the function $\frac{\lambda}{g}$ defined by $\frac{\lambda}{g}(x) = \frac{\lambda}{g(x)}$ is also continuous wherever $g(x) \neq 0$. In particular, the continuity of g implies continuity of $\frac{1}{g}$.

The above theorem can be exploited to generate many continuous functions. They also aid in deciding if certain functions are continuous or not. The following examples illustrate this:

Example 16 Prove that every rational function is continuous.

Solution Recall that every rational function f is given by

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where p and q are polynomial functions. The domain of f is all real numbers except points at which q is zero. Since polynomial functions are continuous (Example 14), f is continuous by (4) of Theorem 1.

Example 17 Discuss the continuity of sine function.

Solution To see this we use the following facts

$$\lim_{x \rightarrow 0} \sin x = 0$$

We have not proved it, but is intuitively clear from the graph of $\sin x$ near 0.

Now, observe that $f(x) = \sin x$ is defined for every real number. Let c be a real number. Put $x = c + h$. If $x \rightarrow c$ we know that $h \rightarrow 0$. Therefore

$$\begin{aligned} \lim_{x \rightarrow c} f(x) &= \lim_{x \rightarrow c} \sin x \\ &= \lim_{h \rightarrow 0} \sin(c + h) \\ &= \lim_{h \rightarrow 0} [\sin c \cos h + \cos c \sin h] \\ &= \lim_{h \rightarrow 0} [\sin c \cos h] + \lim_{h \rightarrow 0} [\cos c \sin h] \\ &= \sin c + 0 = \sin c = f(c) \end{aligned}$$

Thus $\lim_{x \rightarrow c} f(x) = f(c)$ and hence f is a continuous function.

Remark A similar proof may be given for the continuity of cosine function.

Example 18 Prove that the function defined by $f(x) = \tan x$ is a continuous function.

Solution The function $f(x) = \tan x = \frac{\sin x}{\cos x}$. This is defined for all real numbers such

that $\cos x \neq 0$, i.e., $x \neq (2n + 1)\frac{\pi}{2}$. We have just proved that both sine and cosine functions are continuous. Thus $\tan x$ being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if f and g are two real functions, then

$$(f \circ g)(x) = f(g(x))$$

is defined whenever the range of g is a subset of domain of f . The following theorem (stated without proof) captures the continuity of composite functions.

Theorem 2 Suppose f and g are real valued functions such that $(f \circ g)$ is defined at c . If g is continuous at c and if f is continuous at $g(c)$, then $(f \circ g)$ is continuous at c .

The following examples illustrate this theorem.

Example 19 Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

Solution Observe that the function is defined for every real number. The function f may be thought of as a composition $g \circ h$ of the two functions g and h , where $g(x) = \sin x$ and $h(x) = x^2$. Since both g and h are continuous functions, by Theorem 2, it can be deduced that f is a continuous function.

Example 20 Show that the function f defined by

$$f(x) = |1 - x + |x||,$$

where x is any real number, is a continuous function.

Solution Define g by $g(x) = 1 - x + |x|$ and h by $h(x) = |x|$ for all real x . Then

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) \\ &= h(1 - x + |x|) \\ &= |1 - x + |x|| = f(x) \end{aligned}$$

In Example 7, we have seen that h is a continuous function. Hence g being a sum of a polynomial function and the modulus function is continuous. But then f being a composite of two continuous functions is continuous.

EXERCISE 5.1

1. Prove that the function $f(x) = 5x - 3$ is continuous at $x = 0$, at $x = -3$ and at $x = 5$.
2. Examine the continuity of the function $f(x) = 2x^2 - 1$ at $x = 3$.
3. Examine the following functions for continuity.

$$(a) f(x) = x - 5 \qquad (b) f(x) = \frac{1}{x-5}, x \neq 5$$

$$(c) f(x) = \frac{x^2 - 25}{x + 5}, x \neq -5 \qquad (d) f(x) = |x - 5|$$

4. Prove that the function $f(x) = x^n$ is continuous at $x = n$, where n is a positive integer.
5. Is the function f defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 5, & \text{if } x > 1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Find all points of discontinuity of f , where f is defined by

$$6. f(x) = \begin{cases} 2x + 3, & \text{if } x \leq 2 \\ 2x - 3, & \text{if } x > 2 \end{cases} \qquad 7. f(x) = \begin{cases} |x| + 3, & \text{if } x \leq -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x + 2, & \text{if } x \geq 3 \end{cases}$$

$$8. f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \qquad 9. f(x) = \begin{cases} \frac{x}{|x|}, & \text{if } x < 0 \\ -1, & \text{if } x \geq 0 \end{cases}$$

$$10. f(x) = \begin{cases} x + 1, & \text{if } x \geq 1 \\ x^2 + 1, & \text{if } x < 1 \end{cases} \qquad 11. f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

$$12. f(x) = \begin{cases} x^{10} - 1, & \text{if } x \leq 1 \\ x^2, & \text{if } x > 1 \end{cases}$$

13. Is the function defined by

$$f(x) = \begin{cases} x + 5, & \text{if } x \leq 1 \\ x - 5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Discuss the continuity of the function f , where f is defined by

$$14. f(x) = \begin{cases} 3, & \text{if } 0 \leq x \leq 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \leq x \leq 10 \end{cases} \quad 15. f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

$$16. f(x) = \begin{cases} -2, & \text{if } x \leq -1 \\ 2x, & \text{if } -1 < x \leq 1 \\ 2, & \text{if } x > 1 \end{cases}$$

17. Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \leq 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

18. For what value of λ is the function defined by

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \leq 0 \\ 4x+1, & \text{if } x > 0 \end{cases}$$

continuous at $x = 0$? What about continuity at $x = 1$?

19. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x .
20. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$?
21. Discuss the continuity of the following functions:
- (a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$
- (c) $f(x) = \sin x \cdot \cos x$
22. Discuss the continuity of the cosine, cosecant, secant and cotangent functions.
23. Find all points of discontinuity of f , where

$$f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ x+1, & \text{if } x \geq 0 \end{cases}$$

24. Determine if f defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

25. Examine the continuity of f , where f is defined by

$$f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

Find the values of k so that the function f is continuous at the indicated point in Exercises 26 to 29.

$$26. \quad f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & \text{if } x \neq \frac{\pi}{2} \\ 3, & \text{if } x = \frac{\pi}{2} \end{cases} \quad \text{at } x = \frac{\pi}{2}$$

$$27. \quad f(x) = \begin{cases} kx^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases} \quad \text{at } x = 2$$

$$28. \quad f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases} \quad \text{at } x = \pi$$

$$29. \quad f(x) = \begin{cases} kx + 1, & \text{if } x \leq 5 \\ 3x - 5, & \text{if } x > 5 \end{cases} \quad \text{at } x = 5$$

30. Find the values of a and b such that the function defined by

$$f(x) = \begin{cases} 5, & \text{if } x \leq 2 \\ ax + b, & \text{if } 2 < x < 10 \\ 21, & \text{if } x \geq 10 \end{cases}$$

is a continuous function.

31. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

32. Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

33. Examine that $\sin |x|$ is a continuous function.

34. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

5.3. Differentiability

Recall the following facts from previous class. We had defined the derivative of a real function as follows:

Suppose f is a real function and c is a point in its domain. The derivative of f at c is defined by

$$\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

provided this limit exists. Derivative of f at c is denoted by $f'(c)$ or $\frac{d}{dx}(f(x))|_c$. The function defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

wherever the limit exists is defined to be the derivative of f . The derivative of f is denoted by $f'(x)$ or $\frac{d}{dx}(f(x))$ or if $y = f(x)$ by $\frac{dy}{dx}$ or y' . The process of finding derivative of a function is called differentiation. We also use the phrase *differentiate $f(x)$ with respect to x* to mean *find $f'(x)$* .

The following rules were established as a part of algebra of derivatives:

- (1) $(u \pm v)' = u' \pm v'$
- (2) $(uv)' = u'v + uv'$ (Leibnitz or product rule)
- (3) $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, wherever $v \neq 0$ (Quotient rule).

The following table gives a list of derivatives of certain standard functions:

Table 5.3

$f(x)$	x^n	$\sin x$	$\cos x$	$\tan x$
$f'(x)$	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$

Whenever we defined derivative, we had put a caution *provided the limit exists*. Now the natural question is; what if it doesn't? The question is quite pertinent and so is its answer. If $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$ does not exist, we say that f is not differentiable at c .

In other words, we say that a function f is differentiable at a point c in its domain if both

$\lim_{h \rightarrow 0^-} \frac{f(c+h) - f(c)}{h}$ and $\lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$ are finite and equal. A function is said

to be differentiable in an interval $[a, b]$ if it is differentiable at every point of $[a, b]$. As in case of continuity, at the end points a and b , we take the right hand limit and left hand limit, which are nothing but left hand derivative and right hand derivative of the function at a and b respectively. Similarly, a function is said to be differentiable in an interval (a, b) if it is differentiable at every point of (a, b) .

Theorem 3 If a function f is differentiable at a point c , then it is also continuous at that point.

Proof Since f is differentiable at c , we have

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

But for $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c)$$

Therefore
$$\lim_{x \rightarrow c} [f(x) - f(c)] = \lim_{x \rightarrow c} \left[\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right]$$

or
$$\begin{aligned} \lim_{x \rightarrow c} [f(x)] - \lim_{x \rightarrow c} [f(c)] &= \lim_{x \rightarrow c} \left[\frac{f(x) - f(c)}{x - c} \right] \cdot \lim_{x \rightarrow c} [(x - c)] \\ &= f'(c) \cdot 0 = 0 \end{aligned}$$

or
$$\lim_{x \rightarrow c} f(x) = f(c)$$

Hence f is continuous at $x = c$.

Corollary 1 Every differentiable function is continuous.

We remark that the converse of the above statement is not true. Indeed we have seen that the function defined by $f(x) = |x|$ is a continuous function. Consider the left hand limit

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$$

The right hand limit

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1$$

Since the above left and right hand limits at 0 are not equal, $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$ does not exist and hence f is not differentiable at 0. Thus f is not a differentiable function.

5.3.1 Derivatives of composite functions

To study derivative of composite functions, we start with an illustrative example. Say, we want to find the derivative of f , where

$$f(x) = (2x + 1)^3$$

One way is to expand $(2x + 1)^3$ using binomial theorem and find the derivative as a polynomial function as illustrated below.

$$\begin{aligned}\frac{d}{dx} f(x) &= \frac{d}{dx} [(2x+1)^3] \\ &= \frac{d}{dx} (8x^3 + 12x^2 + 6x + 1) \\ &= 24x^2 + 24x + 6 \\ &= 6(2x + 1)^2\end{aligned}$$

Now, observe that $f(x) = (h \circ g)(x)$

where $g(x) = 2x + 1$ and $h(x) = x^3$. Put $t = g(x) = 2x + 1$. Then $f(x) = h(t) = t^3$. Thus

$$\frac{df}{dx} = 6(2x + 1)^2 = 3(2x + 1)^2 \cdot 2 = 3t^2 \cdot 2 = \frac{dh}{dt} \cdot \frac{dt}{dx}$$

The advantage with such observation is that it simplifies the calculation in finding the derivative of, say, $(2x + 1)^{100}$. We may formalise this observation in the following theorem called the chain rule.

Theorem 4 (Chain Rule) Let f be a real valued function which is a composite of two functions u and v ; i.e., $f = v \circ u$. Suppose $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, we have

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

We skip the proof of this theorem. Chain rule may be extended as follows. Suppose f is a real valued function which is a composite of three functions u , v and w ; i.e.,

$f = (w \circ u) \circ v$. If $t = v(x)$ and $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

provided all the derivatives in the statement exist. Reader is invited to formulate chain rule for composite of more functions.

Example 21 Find the derivative of the function given by $f(x) = \sin(x^2)$.

Solution Observe that the given function is a composite of two functions. Indeed, if $t = u(x) = x^2$ and $v(t) = \sin t$, then

$$f(x) = (v \circ u)(x) = v(u(x)) = v(x^2) = \sin x^2$$

Put $t = u(x) = x^2$. Observe that $\frac{dv}{dt} = \cos t$ and $\frac{dt}{dx} = 2x$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = \cos t \cdot 2x$$

It is normal practice to express the final result only in terms of x . Thus

$$\frac{df}{dx} = \cos t \cdot 2x = 2x \cos x^2$$

Alternatively, We can also directly proceed as follows:

$$\begin{aligned} y = \sin(x^2) &\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin x^2) \\ &= \cos x^2 \cdot \frac{d}{dx}(x^2) = 2x \cos x^2 \end{aligned}$$

Example 22 Find the derivative of $\tan(2x + 3)$.

Solution Let $f(x) = \tan(2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \circ u)(x) = v(u(x)) = v(2x + 3) = \tan(2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and

$\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2 \sec^2(2x + 3)$$

Example 23 Differentiate $\sin(\cos(x^2))$ with respect to x .

Solution The function $f(x) = \sin(\cos(x^2))$ is a composition $f(x) = (w \circ v \circ u)(x)$ of the three functions u , v and w , where $u(x) = x^2$, $v(t) = \cos t$ and $w(s) = \sin s$. Put

$t = u(x) = x^2$ and $s = v(t) = \cos t$. Observe that $\frac{dw}{ds} = \cos s$, $\frac{ds}{dt} = -\sin t$ and $\frac{dt}{dx} = 2x$

exist for all real x . Hence by a generalisation of chain rule, we have

$$\frac{df}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx} = (\cos s) \cdot (-\sin t) \cdot (2x) = -2x \sin x^2 \cdot \cos(\cos x^2)$$

Alternatively, we can proceed as follows:

$$y = \sin(\cos x^2)$$

$$\begin{aligned} \text{Therefore } \frac{dy}{dx} &= \frac{d}{dx} \sin(\cos x^2) = \cos(\cos x^2) \frac{d}{dx}(\cos x^2) \\ &= \cos(\cos x^2) (-\sin x^2) \frac{d}{dx}(x^2) \\ &= -\sin x^2 \cos(\cos x^2) (2x) \\ &= -2x \sin x^2 \cos(\cos x^2) \end{aligned}$$

EXERCISE 5.2

Differentiate the functions with respect to x in Exercises 1 to 8.

1. $\sin(x^2 + 5)$ 2. $\cos(\sin x)$ 3. $\sin(ax + b)$
4. $\sec(\tan(\sqrt{x}))$ 5. $\frac{\sin(ax + b)}{\cos(cx + d)}$ 6. $\cos x^3 \cdot \sin^2(x^5)$
7. $2\sqrt{\cot(x^2)}$ 8. $\cos(\sqrt{x})$
9. Prove that the function f given by

$$f(x) = |x - 1|, x \in \mathbf{R}$$

is not differentiable at $x = 1$.

10. Prove that the greatest integer function defined by

$$f(x) = [x], 0 < x < 3$$

is not differentiable at $x = 1$ and $x = 2$.

5.3.2 Derivatives of implicit functions

Until now we have been differentiating various functions given in the form $y = f(x)$. But it is not necessary that functions are always expressed in this form. For example, consider one of the following relationships between x and y :

$$x - y - \pi = 0$$

$$x + \sin xy - y = 0$$

In the first case, we can *solve for* y and rewrite the relationship as $y = x - \pi$. In the second case, it does not seem that there is an easy way to *solve for* y . Nevertheless, there is no doubt about the dependence of y on x in either of the cases. When a relationship between x and y is expressed in a way that it is easy to *solve for* y and write $y = f(x)$, we say that y is given as an *explicit function* of x . In the latter case it

is implicit that y is a function of x and we say that the relationship of the second type, above, gives function *implicitly*. In this subsection, we learn to differentiate implicit functions.

Example 24 Find $\frac{dy}{dx}$ if $x - y = \pi$.

Solution One way is to solve for y and rewrite the above as

$$y = x - \pi$$

But then $\frac{dy}{dx} = 1$

Alternatively, *directly* differentiating the relationship w.r.t., x , we have

$$\frac{d}{dx}(x - y) = \frac{d\pi}{dx}$$

Recall that $\frac{d\pi}{dx}$ means to differentiate the constant function taking value π everywhere w.r.t., x . Thus

$$\frac{d}{dx}(x) - \frac{d}{dx}(y) = 0$$

which implies that

$$\frac{dy}{dx} = \frac{dx}{dx} = 1$$

Example 25 Find $\frac{dy}{dx}$, if $y + \sin y = \cos x$.

Solution We differentiate the relationship directly with respect to x , i.e.,

$$\frac{dy}{dx} + \frac{d}{dx}(\sin y) = \frac{d}{dx}(\cos x)$$

which implies using chain rule

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = -\sin x$$

This gives $\frac{dy}{dx} = -\frac{\sin x}{1 + \cos y}$

where $y \neq (2n + 1)\pi$

5.3.3 Derivatives of inverse trigonometric functions

We remark that inverse trigonometric functions are continuous functions, but we will not prove this. Now we use chain rule to find derivatives of these functions.

Example 26 Find the derivative of f given by $f(x) = \sin^{-1} x$ assuming it exists.

Solution Let $y = \sin^{-1} x$. Then, $x = \sin y$.

Differentiating both sides w.r.t. x , we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$,
i.e., $x \in (-1, 1)$.

To make this result a bit more attractive, we carry out the following manipulation. Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is positive and hence $\cos y = \sqrt{1-x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Example 27 Find the derivative of f given by $f(x) = \tan^{-1} x$ assuming it exists.

Solution Let $y = \tan^{-1} x$. Then, $x = \tan y$.

Differentiating both sides w.r.t. x , we get

$$1 = \sec^2 y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + (\tan(\tan^{-1} x))^2} = \frac{1}{1 + x^2}$$

Finding of the derivatives of other inverse trigonometric functions is left as exercise. The following table gives the derivatives of the remaining inverse trigonometric functions (Table 5.4):

Table 5.4

$f(x)$	$\cos^{-1}x$	$\cot^{-1}x$	$\sec^{-1}x$	$\operatorname{cosec}^{-1}x$
$f'(x)$	$\frac{-1}{\sqrt{1-x^2}}$	$\frac{-1}{1+x^2}$	$\frac{1}{ x \sqrt{x^2-1}}$	$\frac{-1}{ x \sqrt{x^2-1}}$
Domain of f'	$(-1, 1)$	R	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, -1) \cup (1, \infty)$

EXERCISE 5.3

Find $\frac{dy}{dx}$ in the following:

1. $2x + 3y = \sin x$
2. $2x + 3y = \sin y$
3. $ax + by^2 = \cos y$
4. $xy + y^2 = \tan x + y$
5. $x^2 + xy + y^2 = 100$
6. $x^3 + x^2y + xy^2 + y^3 = 81$
7. $\sin^2 y + \cos xy = \kappa$
8. $\sin^2 x + \cos^2 y = 1$
9. $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$
10. $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$
11. $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$
12. $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$
13. $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$
14. $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$
15. $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$