Evaluate $\int \frac{\sin x}{\sin 4x} dx$.

Answer: $I = \int \frac{\sin x}{\sin 4x} dx = \int \frac{\sin x}{2\sin 2x\cos 2x} dx$

$$= \int \frac{\sin x}{4\sin x\cos x\cos 2x} dx$$

$$= \frac{1}{4}\int \frac{1}{\cos x\cos 2x} dx = \frac{1}{4}\int \frac{\cos x}{\cos^2 x\cos 2x} dx$$

$$= \frac{1}{4}\int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)} dx$$

Putting $\sin x = u$ and $\cos x dx = du$, we get

$$\begin{aligned} I &= \frac{1}{4}\int \frac{du}{(1-u^2)(1-2u^2)} \\ &= \frac{1}{4}\int \left[\frac{2}{1-2u^2} - \frac{1}{1-u^2}\right] du \\ &= -\frac{1}{4}\int \frac{1}{1-u^2} du + \frac{2}{4}\int \frac{1}{1-(\sqrt{2}u)^2} du \\ &= -\frac{1}{4}\times\frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + \frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}u}{1-\sqrt{2}u}\right| + C \\ &= -\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C \end{aligned}$$