Appendix 1

INFINITE SERIES

A.1.1 Introduction

As discussed in the Chapter 9 on Sequences and Series, a sequence $a_1, a_2, ..., a_n, ...$ having infinite number of terms is called *infinite sequence* and its indicated sum, i.e., $a_1 + a_2 + a_3 + ... + a_n + ...$ is called an *infinte series* associated with infinite sequence. This series can also be expressed in abbreviated form using the sigma notation, i.e.,

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = \sum_{k=1}^{\infty} a_k$$

In this Chapter, we shall study about some special types of series which may be required in different problem situations.

A.1.2 Binomial Theorem for any Index

In Chapter 8, we discussed the Binomial Theorem in which the index was a positive integer. In this Section, we state a more general form of the theorem in which the index is not necessarily a whole number. It gives us a particular type of infinite series, called *Binomial Series*. We illustrate few applications, by examples.

We know the formula

$$(1 + x)^n = {}^nC_0 + {}^nC_1\, x + \ldots + {}^nC_n\, x^n$$

Here, n is non-negative integer. Observe that if we replace index n by negative integer or a fraction, then the combinations ${}^nC_r$ do not make any sense.

We now state (without proof), the Binomial Theorem, giving an infinite series in which the index is negative or a fraction and not a whole number.

Theorem The formula

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \ldots$$

holds whenever $|x| < 1$.

Remark 1. Note carefully the condition $|x| < 1$, i.e., $-1 < x < 1$ is necessary when m is negative integer or a fraction. For example, if we take $x = -2$ and $m = -2$, we obtain

$$(1-2)^{-2} = 1+(-2)(-2)+\frac{(-2)(-3)}{1.2}(-2)^2+...$$

or $\quad 1 = 1 + 4 + 12 + \ldots$

This is not possible

2. Note that there are infinite number of terms in the expansion of $(1+x)^m$, when m is a negative integer or a fraction

Consider $$(a+b)^m = \left[a\left(1+\frac{b}{a}\right)\right]^m = a^m\left(1+\frac{b}{a}\right)^m$$

$$= a^m\left[1+m\frac{b}{a}+\frac{m(m-1)}{1.2}\left(\frac{b}{a}\right)^2+...\right]$$

$$= a^m + ma^{m-1}b+\frac{m(m-1)}{1.2}a^{m-2}b^2+...$$

This expansion is valid when $\left|\frac{b}{a}\right| < 1$ or equivalently when $|b| < |a|$.

The general term in the expansion of $(a + b)^m$ is

$$\frac{m(m-1)(m-2)...(m-r+1)a^{m-r}b^r}{1.2.3...r}$$

We give below certain particular cases of Binomial Theorem, when we assume $|x| < 1$, these are left to students as exercises:

1. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots$
2. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$
3. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$
4. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

Example 1 Expand $\left(1-\frac{x}{2}\right)^{-\frac{1}{2}}$, when $|x| < 2$.

Solution We have

$$\left(1-\frac{x}{2}\right)^{-\frac{1}{2}} = 1+\frac{\left(-\frac{1}{2}\right)}{1}\left(\frac{-x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2}\left(-\frac{x}{2}\right)^{2}+...$$

$$= 1+\frac{x}{4}+\frac{3x^{2}}{32}+...$$

A.1.3 Infinite Geometric Series

From Chapter 9, Section 9.5, a sequence a_1, a_2, a_3, ..., a_n is called G.P., if $\frac{a_{k+1}}{a_k}$ = r (constant) for k = 1, 2, 3, ..., n–1. Particularly, if we take $a_1 = a$, then the resulting sequence $a, ar, ar^2, ..., ar^{n-1}$ is taken as the standard form of G.P., where a is first term and r, the common ratio of G.P.

Earlier, we have discussed the formula to find the sum of finite series $a + ar + ar^2 + ... + ar^{n-1}$ which is given by

$$S_n = \frac{a\left(1-r^n\right)}{1-r}.$$

In this section, we state the formula to find the sum of infinite geometric series $a + ar + ar^2 + ... + ar^{n-1} + ...$ and illustrate the same by examples.

Let us consider the G.P. $1, \frac{2}{3}, \frac{4}{9},...$

Here $a = 1$, $r = \frac{2}{3}$. We have

$$S_n = \frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}} = 3\left[1-\left(\frac{2}{3}\right)^n\right] \quad ...(1)$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger.