8.2 Solved Examples

Short Answer Type

Example 1 Find the r^{th} term in the expansion of $\left(x+\frac{1}{x}\right)^{2r}$.

Solution We have $T_r = {}^{2r}C_{r-1}\ (x)^{2r-r+1}\left(\frac{1}{x}\right)^{r-1}$

$$= \frac{\lfloor 2r}{\lfloor r-1\ \lfloor r+1}\ x^{r+1-r+1}$$

$$= \frac{\lfloor 2r}{\lfloor r-1\ \lfloor r+1}\ x^{2}$$

Example 2 Expand the following $(1-x+x^2)^4$

Solution Put $1-x = y$. Then

$$\begin{aligned}(1-x+x^2)^4 &= (y+x^2)^4\\ &= {}^4C_0\ y^4(x^2)^0 + {}^4C_1\ y^3(x^2)^1\\ &\quad + {}^4C_2\ y^2(x^2)^2 + {}^4C_3\ y(x^2)^3 + {}^4C_4\ (x^2)^4\\ &= y^4 + 4y^3x^2 + 6y^2x^4 + 4yx^6 + x^8\\ &= (1-x)^4 + 4x^2(1-x)^3 + 6x^4(1-x)^2 + 4x^6(1-x) + x^8\\ &= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8\end{aligned}$$

Example 3 Find the 4^{th} term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^9$

Solution Since r^{th} term from the end in the expansion of $(a+b)^n$ is $(n-r+2)^{th}$ term from the beginning. Therefore 4^{th} term from the end is $9-4+2$, i.e., 7^{th} term from the beginning, which is given by

$$T_7 = {}^9C_6\left(\frac{x^3}{2}\right)^3\left(\frac{-2}{x^2}\right)^6 = {}^9C_3\,\frac{x^9}{8}\cdot\frac{64}{x^{12}} = \frac{9\times8\times7}{3\times2\times1}\times\frac{64}{x^3} = \frac{672}{x^3}$$

Example 4 Evaluate: $\left(x^2-\sqrt{1-x^2}\right)^4 + \left(x^2+\sqrt{1-x^2}\right)^4$

Solution Putting $\sqrt{1-x^2} = y$, we get

The given expression $= (x^2 - y)^4 + (x^2 + y)^4 = 2\ [x^8 + {}^4C_2\, x^4\, y^2 + {}^4C_4\, y^4]$

$$= 2\left[x^8 + \frac{4\times 3}{2\times 1}x^4\cdot(1-x^2)+(1-x^2)^2\right]$$

$$= 2\ [x^8 + 6x^4\ (1 - x^2) + (1 - 2x^2 + x^4]$$

$$= 2x^8 - 12x^6 + 14x^4 - 4x^2 + 2$$

Example 5 Find the coefficient of x^{11} in the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{12}$

Solution Let the general term, i.e., $(r + 1)^{th}$ contain x^{11}.

We have

$$T_{r+1} = {}^{12}C_r\ (x^3)^{12-r}\left(-\frac{2}{x^2}\right)^r$$

$$= {}^{12}C_r\ x^{36-3r-2r}\ (-1)^r\ 2^r$$

$$= {}^{12}C_r\ (-1)^r\ 2^r\ x^{36-5r}$$

Now for this to contain x^{11}, we observe that

$$36 - 5r = 11, \text{ i.e., } r = 5$$

Thus, the coefficient of x^{11} is

$${}^{12}C_5\ (-1)^5\ 2^5 = -\frac{12\times 11\times 10\times 9\times 8}{5\times 4\times 3\times 2}\times 32 = -25344$$

Example 6 Determine whether the expansion of $\left(x^2 - \frac{2}{x}\right)^{18}$ will contain a term containing x^{10}?

Solution Let T_{r+1} contain x^{10}. Then

$$T_{r+1} = {}^{18}C_r\,(x^2)^{18-r}\left(\frac{-2}{x}\right)^r$$

$$= {}^{18}C_r\ x^{36-2r}\ (-1)^r\ .\ 2^r\ x^{-r}$$

$$= (-1)^r\ 2^r\ {}^{18}C_r\ x^{36-3r}$$

Thus, $36 - 3r = 10, \text{ i.e., } r = \frac{26}{3}$

Since r is a fraction, the given expansion cannot have a term containing x^{10}.

Example 7 Find the term independent of x in the expansion of $\left(\frac{\sqrt{x}}{\sqrt{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$.

Solution Let $(r+1)^{\text{th}}$ term be independent of x which is given by

$$T_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \left(\frac{\sqrt{3}}{2x^2}\right)^r$$

$$= {}^{10}C_r \left(\frac{x}{3}\right)^{\frac{10-r}{2}} 3^{\frac{r}{2}} \left(\frac{1}{2^r x^{2r}}\right)$$

$$= {}^{10}C_r\, 3^{\frac{r}{2}-\frac{10-r}{2}}\, 2^{-r}\, x^{\frac{10-r}{2}-2r}$$

Since the term is independent of x, we have

$$\frac{10-r}{2} - 2r = 0 \quad \Rightarrow \quad r = 2$$

Hence 3^{rd} term is independent of x and its value is given by

$$T_3 = {}^{10}C_2 \frac{3^{-3}}{4} = \frac{10\times 9}{2\times 1}\times\frac{1}{9\times 12} = \frac{5}{12}$$

Example 8 Find the middle term in the expansion of $\left(2ax - \frac{b}{x^2}\right)^{12}$.

Solution Since the power of binomial is even, it has one middle term which is the

$\left(\frac{12+2}{2}\right)^{\text{th}}$ term and it is given by

$$T_7 = {}^{12}C_6 (2ax)^6 \left(\frac{-b}{x^2}\right)^6$$

$$= {}^{12}C_6 \frac{2^6 a^6 x^6 \cdot (-b)^6}{x^{12}}$$

$$= {}^{12}C_6 \frac{2^6 a^6 b^6}{x^6} = \frac{59136 a^6 b^6}{x^6}$$

Example 9 Find the middle term (terms) in the expansion of $\left(\frac{p}{x}+\frac{x}{p}\right)^9$.

Solution Since the power of binomial is odd. Therefore, we have two middle terms which are 5th and 6th terms. These are given by

$$T_5 = {}^9C_4\left(\frac{p}{x}\right)^5\left(\frac{x}{p}\right)^4 = {}^9C_4\frac{p}{x} = \frac{126p}{x}$$

and
$$T_6 = {}^9C_5\left(\frac{p}{x}\right)^4\left(\frac{x}{p}\right)^5 = {}^9C_5\frac{x}{p} = \frac{126x}{p}$$

Example 10 Show that $2^{4n+4} - 15n - 16$, where $n \in \mathbf{N}$ is divisible by 225.

Solution We have

$$\begin{aligned}
2^{4n+4} - 15n - 16 &= 2^{4(n+1)} - 15n - 16 \\
&= 16^{n+1} - 15n - 16 \\
&= (1+15)^{n+1} - 15n - 16 \\
&= {}^{n+1}C_0\, 15^0 + {}^{n+1}C_1\, 15^1 + {}^{n+1}C_2\, 15^2 + {}^{n+1}C_3\, 15^3 \\
&\quad + \dots + {}^{n+1}C_{n+1}\, (15)^{n+1} - 15n - 16 \\
&= 1 + (n+1)\, 15 + {}^{n+1}C_2\, 15^2 + {}^{n+1}C_3\, 15^3 \\
&\quad + \dots + {}^{n+1}C_{n+1}\, (15)^{n+1} - 15n - 16 \\
&= 1 + 15n + 15 + {}^{n+1}C_2\, 15^2 + {}^{n+1}C_3\, 15^3 \\
&\quad + \dots + {}^{n+1}C_{n+1}\, (15)^{n+1} - 15n - 16 \\
&= 15^2\, [{}^{n+1}C_2 + {}^{n+1}C_3\, 15 + \dots \text{ so on}]
\end{aligned}$$

Thus, $2^{4n+4} - 15n - 16$ is divisible by 225.

Long Answer Type

Example 11 Find numerically the greatest term in the expansion of $(2 + 3x)^9$, where $x = \frac{3}{2}$.

Solution We have $(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9$

Now, $$\frac{T_{r+1}}{T_r} = \frac{2^9\left[{}^9C_r\left(\frac{3x}{2}\right)^r\right]}{2^9\left[{}^9C_{r-1}\left(\frac{3x}{2}\right)^{r-1}\right]}$$

$$= \frac{{}^9C_r}{{}^9C_{r-1}}\left|\frac{3x}{2}\right| = \frac{9!}{r!\,(9-r)!}\cdot\frac{(r-1)!\,(10-r)!}{9!}\left|\frac{3x}{2}\right|$$

$$= \frac{10-r}{r}\left|\frac{3x}{2}\right| = \frac{10-r}{r}\left(\frac{9}{4}\right) \qquad \text{Since } x = \frac{3}{2}$$

Therefore, $$\frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{90-9r}{4r} \geq 1$$

$$\Rightarrow 90 - 9r \geq 4r \qquad \text{(Why?)}$$

$$\Rightarrow r \leq \frac{90}{13}$$

$$\Rightarrow r \leq 6\frac{12}{13}$$

Thus the maximum value of r is 6. Therefore, the greatest term is $T_{r+1} = T_7$.

Hence, $$T_7 = 2^9\left[{}^9C_6\left(\frac{3x}{2}\right)^6\right], \qquad \text{where } x = \frac{3}{2}$$

$$= 2^9 \cdot {}^9C_6\left(\frac{9}{4}\right)^6 = 2^9 \cdot \frac{9\times 8\times 7}{3\times 2\times 1}\left(\frac{3^{12}}{2^{12}}\right) = \frac{7\times 3^{13}}{2}$$

Example 12 If n is a positive integer, find the coefficient of x^{-1} in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$.

Solution We have

$$(1+x)^n\left(1+\frac{1}{x}\right)^n = (1+x)^n\left(\frac{x+1}{x}\right)^n = \frac{(1+x)^{2n}}{x^n}$$

Now to find the coefficient of x^{-1} in $(1+x)^n\left(1+\frac{1}{x}\right)^n$, it is equivalent to finding coefficient of x^{-1} in $\frac{(1+x)^{2n}}{x^n}$ which in turn is equal to the coefficient of x^{n-1} in the expansion of $(1+x)^{2n}$.

Since $(1+x)^{2n} = {}^{2n}C_0\, x^0 + {}^{2n}C_1\, x^1 + {}^{2n}C_2\, x^2 + \ldots + {}^{2n}C_{n-1}\, x^{n-1} + \ldots + {}^{2n}C_{2n}\, x^{2n}$

Thus the coefficient of x^{n-1} is ${}^{2n}C_{n-1}$

$$= \frac{\lfloor 2n}{\lfloor n-1\,\lfloor 2n-n+1} = \frac{\lfloor 2n}{\lfloor n-1\,\lfloor n+1}$$

Example 13 Which of the following is larger?

$99^{50} + 100^{50}$ or 101^{50}

We have $(101)^{50} = (100+1)^{50}$

$$= 100^{50} + 50\,(100)^{49} + \frac{50.49}{2.1}(100)^{48} + \frac{50.49.48}{3.2.1}(100)^{47} + \ldots \quad (1)$$

Similarly $\quad 99^{50} = (100-1)^{50}$

$$= 100^{50} - 50\,.\,100^{49} + \frac{50.49}{2.1}(100)^{48} - \frac{50.49.48}{3.2.1}(100)^{47} + \ldots \quad (2)$$

Subtracting (2) from (1), we get

$$101^{50} - 99^{50} = 2\left[50\cdot(100)^{49} + \frac{50\cdot49\cdot48}{3\cdot2\cdot1}100^{47} + \ldots\right]$$

$$\Rightarrow \quad 101^{50} - 99^{50} = 100^{50} + 2\left[\frac{50\cdot49\cdot48}{3\cdot2\cdot1}100^{47} + \ldots\right]$$

$\Rightarrow \quad 101^{50} - 99^{50} > 100^{50}$

Hence $101^{50} > 99^{50} + 100^{50}$

Example 14 Find the coefficient of x^{50} after simplifying and collecting the like terms in the expansion of $(1+x)^{1000} + x\,(1+x)^{999} + x^2\,(1+x)^{998} + \ldots + x^{1000}$.

Solution Since the above series is a geometric series with the common ratio $\frac{x}{1+x}$, its sum is

$$\frac{(1+x)^{1000}\ 1-\ \frac{x}{1+x}^{1001}}{1-\ \frac{x}{1+x}}$$

$$=\frac{(1+x)^{1000}-\frac{x^{1001}}{1+x}}{\frac{1+x-x}{1+x}}=(1+x)^{1001}-x^{1001}$$

Hence, coefficient of x^{50} is given by

$${}^{1001}C_{50}=\frac{\lfloor 1001}{\lfloor 50\,\lfloor 951}$$

Example 15 If a_1, a_2, a_3 and a_4 are the coefficient of any four consecutive terms in the expansion of $(1+x)^n$, prove that

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$$

Solution Let a_1, a_2, a_3 and a_4 be the coefficient of four consecutive terms T_{r+1}, T_{r+2}, T_{r+3}, and T_{r+4} respectively. Then

$$a_1 = \text{coefficient of } T_{r+1} = {}^nC_r$$

$$a_2 = \text{coefficient of } T_{r+2} = {}^nC_{r+1}$$

$$a_3 = \text{coefficient of } T_{r+3} = {}^nC_{r+2}$$

and $$a_4 = \text{coefficient of } T_{r+4} = {}^nC_{r+3}$$

Thus $$\frac{a_1}{a_1+a_2}=\frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}$$

$$=\frac{{}^nC_r}{{}^{n+1}C_{r+1}}\quad (\because\ {}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1})$$

$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \times \frac{\underline{|r+1}\,\underline{|n-r}}{\underline{|n+1}} = \frac{r+1}{n+1}$$

Similarly, $$\frac{a_3}{a_3+a_4} = \frac{{}^nC_{r+2}}{{}^nC_{r+2}+{}^nC_{r+3}}$$

$$= \frac{{}^nC_{r+2}}{{}^{n+1}C_{r+3}} = \frac{r+3}{n+1}$$

Hence, $$\text{L.H.S.} = \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2r+4}{n+1}$$

and $$\text{R.H.S.} = \frac{2a_2}{a_2+a_3} = \frac{2\left({}^nC_{r+1}\right)}{{}^nC_{r+1}+{}^nC_{r+2}} = \frac{2\left({}^nC_{r+1}\right)}{{}^{n+1}C_{r+2}}$$

$$= 2\frac{\underline{|n}}{\underline{|r+1}\,\underline{|n-r-1}} \times \frac{\underline{|r+2}\,\underline{|n-r-1}}{\underline{|n+1}} = \frac{2(r+2)}{n+1}$$

Objective Type Questions (M.C.Q)

Example 16 The total number of terms in the expansion of $(x+a)^{51} - (x-a)^{51}$ after simplification is

(a) 102 (b) 25 (c) 26 (d) None of these

Solution C is the correct choice since the total number of terms are 52 of which 26 terms get cancelled.

Example 17 If the coefficients of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55 (c) 45 (d) 15

Solution B is the correct choice. Since $T_{r+1} = {}^nC_r\, a^{n-r}\, x^r$ in expansion of $(a+x)^n$,

Therefore, $$T_8 = {}^nC_7 (2)^{n-7}\left(\frac{x}{3}\right)^7 = {}^nC_7\, \frac{2^{n-7}}{3^7}\, x^7$$

and $$T_9 = {}^nC_8\,(2)^{n-8}\left(\frac{x}{3}\right)^8 = {}^nC_8\,\frac{2^{n-8}}{3^8}\,x^8$$

Therefore, $${}^nC_7\,\frac{2^{n-7}}{3^7} = {}^nC_8\,\frac{2^{n-8}}{3^8}$$ (since it is given that coefficient of x^7 = coefficient x^8)

$$\Rightarrow \qquad \frac{\lfloor n}{\lfloor 7\lfloor n-7} \times \frac{\lfloor 8\lfloor n-8}{\lfloor n} = \frac{2^{n-8}}{3^8}\cdot\frac{3^7}{2^{n-7}}$$

$$\Rightarrow \qquad \frac{8}{n-7} = \frac{1}{6} \quad \Rightarrow \quad n = 55$$

Example 18 If $(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$, then $a_0 + a_2 + a_4 + ... + a_{2n}$ equals.

(A) $\frac{3^n+1}{2}$ (B) $\frac{3^n-1}{2}$ (C) $\frac{1-3^n}{2}$ (D) $3^n + \frac{1}{2}$

Solution A is the correct choice. Putting $x = 1$ and -1 in

$$(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$$

we get $$1 = a_0 + a_1 + a_2 + a_3 + ... + a_{2n} \qquad ... (1)$$

and $$3^n = a_0 - a_1 + a_2 - a_3 + ... + a_{2n} \qquad ... (2)$$

Adding (1) and (2), we get

$$3^n + 1 = 2\,(a_0 + a_2 + a_4 + ... + a_{2n})$$

Therefore $$a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n+1}{2}$$

Example 19 The coefficient of x^p and x^q (p and q are positive integers) in the expansion of $(1 + x)^{p+q}$ are

(A) equal (B) equal with opposite signs

(C) reciprocal of each other (D) none of these

Solution A is the correct choice. Coefficient of x^p and x^q in the expansion of $(1 + x)^{p+q}$ are ${}^{p+q}C_p$ and ${}^{p+q}C_q$

and $${}^{p+q}C_p = {}^{p+q}C_q = \frac{\lfloor p+q}{\lfloor p\lfloor q}$$

Hence (a) is the correct answer.

Example 20 The number of terms in the expansion of $(a + b + c)^n$, where $n \in \mathbf{N}$ is

(A) $\frac{(n+1)(n+2)}{2}$ (B) $n + 1$

(C) $n + 2$ (D) $(n + 1)\, n$

Solution A is the correct choice. We have

$$(a + b + c)^n = [a + (b + c)]^n$$
$$= a^n + {}^nC_1\, a^{n-1} (b + c)^1 + {}^nC_2\, a^{n-2} (b + c)^2 + ... + {}^nC_n (b + c)^n$$

Further, expanding each term of R.H.S., we note that

First term consist of 1 term.

Second term on simplification gives 2 terms.

Third term on expansion gives 3 terms.

Similarly, fourth term on expansion gives 4 terms and so on.

$$\text{The total number of terms} = 1 + 2 + 3 + ... + (n + 1)$$
$$= \frac{(n+1)(n+2)}{2}$$

Example 21 The ratio of the coefficient of x^{15} to the term independent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$ is

(A) 12:32 (B) 1:32 (C) 32:12 (D) 32:1

Solution (B) is the correct choice. Let T_{r+1} be the general term of $\left(x^2 + \frac{2}{x}\right)^{15}$, so,

$$T_{r+1} = {}^{15}C_r\, (x^2)^{15-r} \left(\frac{2}{x}\right)^r$$

$$= {}^{15}C_r\, (2)^r\, x^{30-3r} \quad ... (1)$$

Now, for the coefficient of term containing x^{15},

$$30 - 3r = 15, \quad \text{i.e.,} \quad r = 5$$

Therefore, ${}^{15}C_5\, (2)^5$ is the coefficient of x^{15} (from (1))

To find the term independent of x, put $30 - 3r = 0$