5) Solve $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$ given that $y(0)=\sqrt{5}$.

Solution: $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$

Solving quadratic in $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{-2x \pm \sqrt{4x^2+4y^2}}{2y}$$

$$= \frac{-x \pm \sqrt{x^2+y^2}}{y}$$ which is homogenous..

Put $y = vx$, i.e. $\frac{dy}{dx} = v + x\frac{dy}{dx}$

Then given equation transforms to

$$v + x\frac{dv}{dx} = \frac{-1 \pm \sqrt{1+v^2}}{v}$$

or $$v^2 + xv\frac{dv}{dx} = -1 \pm \sqrt{1+v^2}$$

or $$(v^2+1) \pm \sqrt{1+v^2} = -xv\frac{dv}{dx}$$

or $$\int \frac{v\,dv}{(1+v^2) \pm \sqrt{v^2+1}} = -\int \frac{dx}{x}$$

or $$\int \frac{v\,dv}{\sqrt{v^2+1}\left(\sqrt{1+v^2} \pm 1\right)} = -\int \frac{dx}{x} \quad \text{——— (1)}$$

Put $\sqrt{1+v^2} \pm 1 = t$, i.e. $\frac{v}{\sqrt{1+v^2}}\,dv = dt$

Then equation (1) transforms to $\int \frac{dt}{t} = -\int \frac{dx}{x}$

or $\ln t = -\ln x + \ln c$

or $tx = c$

or $\left(\sqrt{1+v^2} \pm 1\right)x = c$

or $\sqrt{x^2+y^2} \pm x = c$

Given when $x=0$, $y=\sqrt{5}$

or $[\sqrt{5} - 0] = c$ or $c = \sqrt{5}$

$\therefore \sqrt{x^2+y^2} = \sqrt{5} \pm x$

or $x^2+y^2 = 5+x^2 \pm 2\sqrt{5}x \Rightarrow y^2 = 5 \pm 2\sqrt{5}x$