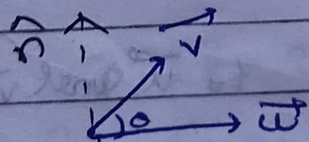
Vector Product (cross-Product) of two vectors

suppose $\vec{u}$ and $\vec{v}$ are non-collinear vectors. The cross product of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \times \vec{v}$, is defined by

$$\vec{u} \times \vec{v} = |\vec{u}|\,|\vec{v}| \sin\theta \; \hat{n}$$

where $\theta = \vec{u} \wedge \vec{v}$, $\vec{u}, \vec{v}, \hat{n}$ form a right handed system



If $\vec{u}$ & $\vec{v}$ are collinear, we define

$$\vec{u} \times \vec{v} = \vec{0}$$

$$\vec{0} \times \vec{0} = \vec{0}, \quad \vec{0} \times \vec{v} = \vec{0}, \quad \vec{u} \times \vec{0} = \vec{0}$$

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If $\vec{u}$ and $\vec{v}$ are non-collinear

$$\vec{u} \times \vec{v} = \underbrace{|\vec{u}|.|\vec{v}| \sin\theta}_{\text{magnitude of } \vec{u} \times \vec{v}} \hat{n}$$

$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

dir'n of $\vec{u} \times \vec{v}$: Normal to plane of $\vec{u}$ & $\vec{v}$ and such that $\vec{u}, \vec{v}, \hat{n}$ form a right handed sys

$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$

$\therefore |\vec{v} \times \vec{u}| = |\vec{u} \times \vec{v}|$

and direction exactly opposite.

$|\vec{v} \times \vec{u}| = |\vec{v}||\vec{u}| \sin\theta = |\vec{u}||\vec{v}| \sin\theta = |\vec{u} \times \vec{v}|$

$\hat{n} = \dfrac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$ (unit vector along $\vec{u} \times \vec{v}$)

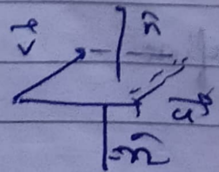
Unit vector ~~to~~ normal to plane of $\vec{u}$ & $\vec{v}$

~~A unit vector norm~~

A vector normal to $\vec{u}$ and $\vec{v}$ = $\vec{u} \times \vec{v}$

Any vector normal to $\vec{u}$ and $\vec{v}$ = $t(\vec{u} \times \vec{v})$, $t \in R$

Unit vectors normal to $\vec{u}$ and $\vec{v}$ = $\pm \dfrac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$



$\vec{u} \times \vec{v}$ is normal to $\vec{u}$ and $\vec{v}$

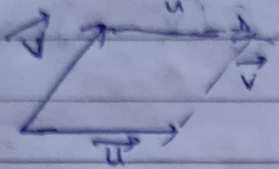
$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$

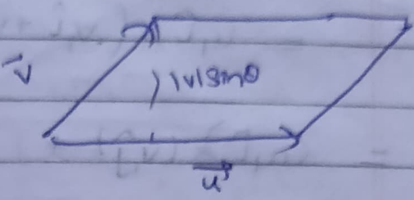
$\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$

$\vec{u}\cdot(\vec{u}\times\vec{v})=0$, $\quad \vec{v}\cdot(\vec{u}\times\vec{v})=0$

$(\because \vec{u}\perp \vec{u}\times\vec{v})$ $\quad (\because \vec{v}\perp \vec{u}\times\vec{v})$

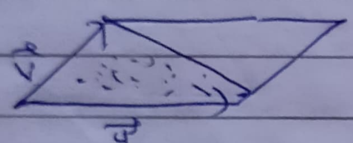
Area of ||gm





<u>Geometrical Area</u> $= |\vec{u}\times\vec{v}|$

$$\text{Area} = |\vec{u}|\,(|\vec{v}|\sin\theta)$$
$$= |\vec{u}|\,|\vec{v}|\sin\theta$$
$$= |\vec{u}\times\vec{v}|$$
$$\cancel{= |\vec{v}\times\vec{u}|}$$



$$\text{Area} = \frac{1}{2}|\vec{u}\times\vec{v}|$$

Algebraic Area $= \vec{u}\times\vec{v}$ $\quad (\vec{v}\times\vec{u})$

<u>Properties</u>

$$\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$$

$$\cancel{\lambda(\vec{u}\times\vec{v}) = \lambda\vec{u}\times\lambda\vec{v}}$$

$$(\lambda\vec{u})\times\vec{v} = \lambda(\vec{u}\times\vec{v}) = \vec{u}\times(\lambda\vec{v})$$

$$\hat{i}\times\hat{i}=\vec{0},\quad \hat{j}\times\hat{j}=\vec{0},\quad \hat{k}\times\hat{k}=\vec{0}$$

$$\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$$
$$\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j}$$

$$\hat{i}\times\hat{j} = |\hat{i}||\hat{j}|\hat{k} = |1||1|\sin\frac{\pi}{2}\,\hat{k} = \hat{k}$$

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$$\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$$
$$\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$$

$$\vec{u}\times\vec{v} = (u_1\hat{i} + u_2\hat{j} + u_3\hat{k})(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$
$$= u_1\hat{i}\times(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) + u_2\hat{j}\times(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}) + u_3\hat{k}\times(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$

$$= (u_1\hat{i})\times(v_1\hat{j}) + \ldots$$

$$= (u_2v_3 - u_3v_2)\hat{i} + (u_3v_1 - u_1v_3)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{u}\times\vec{v} = (u_1\hat{i} + u_2\hat{j} + u_3\hat{k})(v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$
$$= (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

$$2\hat{i}\times(\hat{i} + \hat{j} - \hat{k}) = 2\hat{k} + 2\hat{j}$$

$$\hat{i}\times(\hat{j} + \hat{k}) = \hat{k} - \hat{j}$$

$$\vec{u}\times(\vec{v} + \vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$$
$$\vec{u}\times(\vec{v} + \vec{w} + \vec{x}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w} + \vec{u}\times\vec{x}$$
$$(\vec{u} + \vec{v})\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$$

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$\vec{u} \times (\vec{v}+\vec{w}) = \vec{u} \times (\vec{w} \times \vec{v}) = \vec{u}\times\vec{w} + \vec{u}\times\vec{v}$

Distrib...

$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$ (anti commutative)

Eg Let $\vec{u} = 4\hat{i} - \hat{j} + 3\hat{k}$

$\vec{v} = -2\hat{i} + \hat{j} - 2\hat{k}$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = -\hat{i} + 2\hat{j} + 2\hat{k}$$

$$= \hat{i}\begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 4 & 3 \\ -2 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 4 & -1 \\ -2 & 1 \end{vmatrix}$$

$(\vec{u}\times\vec{v})\cdot\vec{u} = (-1)(4) + 2(-1) + 2(3) = -4-2+6 = 0$

$(\vec{u}\times\vec{v})\cdot\vec{v} = 0$

Area of the ||gm formed by vectors $\vec{u}$ and $\vec{v}$

$= |\vec{u}\times\vec{v}| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = 3$

Area of the triangle formed by vectors $\vec{u}$ and $\vec{v}$ (as its adjacent sides)

$= \frac{1}{2}|\vec{u}\times\vec{v}| = 3/2$

Unit vector normal to $\vec{u}$ & $\vec{v}$ $= \pm \dfrac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|}$

$= \pm \left(\dfrac{-\hat{i}+2\hat{j}+2\hat{k}}{3}\right)$

vector of magnitude 12 which is $\perp r$ to $\vec{u}$ & $\vec{v}$ $= 12\hat{n} = \pm 4(-\hat{i}+2\hat{j}+2\hat{k})$

Alt.

(collinear to $\vec{u}\times\vec{v}$)

$= t(\vec{u}\times\vec{v}) = t(-\hat{i}+2\hat{j}+2\hat{k})$

$|t|3 = 12 \Rightarrow |t| = 4 \Rightarrow t = \pm 4$

$\therefore$ Req. vector $= \pm 4(-\hat{i}+2\hat{j}+2\hat{k})$

Req. vector = 12 $\left(\frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} \right)$

Some properties

1. $(\vec{u} \cdot \vec{v})^2 + |\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2$

$|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$

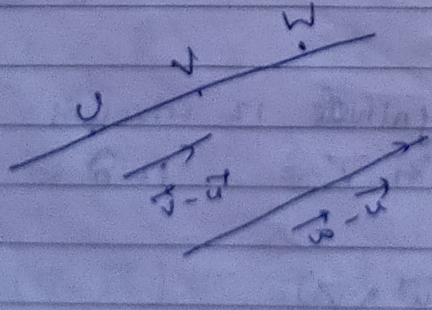
Proof: $(\vec{u} \cdot \vec{v})^2 + |\vec{u} \times \vec{v}|^2 = (|\vec{u}| |\vec{v}| \cos\theta)^2 + (|\vec{u}| |\vec{v}| \sin\theta)^2$
$= |\vec{u}|^2 |\vec{v}|^2 (\cos^2\theta + \sin^2\theta)$
$= |\vec{u}|^2 |\vec{v}|^2$

2. $\vec{u} \times \vec{v} + \vec{v} \times \vec{w} + \vec{w} \times \vec{u} = (\vec{u} - \vec{w}) \times (\vec{v} - \vec{w})$
$= (\vec{w} - \vec{v}) \times (\vec{u} - \vec{v})$
$= (\vec{v} - \vec{u}) \times (\vec{w} - \vec{u})$

$(\vec{u} - \vec{w}) \times (\vec{v} - \vec{w}) = \vec{u} \times (\vec{v} - \vec{w}) - \vec{w} \times (\vec{v} - \vec{w})$
$= \vec{u} \times \vec{v} - \vec{u} \times \vec{w} - \vec{w} \times \vec{v} + \vec{w} \times \vec{w}$
$= \vec{u} \times \vec{w} + \vec{w} \times \vec{u} + \vec{v} \times \vec{w} + \vec{0}$
$= \vec{u} \times \vec{v} + \vec{v} \times \vec{w} + \vec{w} \times \vec{u}$

3. Consider 3 points with p.v $\vec{u}, \vec{v}, \vec{w}$

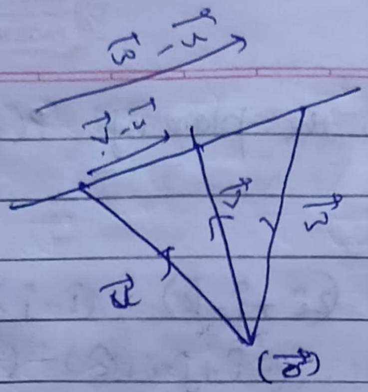
$U(\vec{u})$, $V(\vec{v})$, $W(\vec{w})$



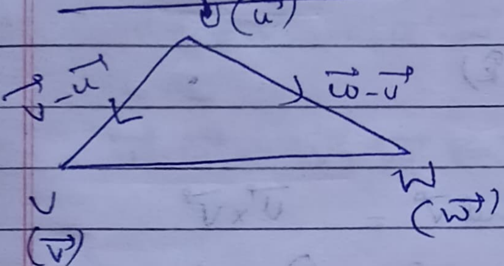
$\vec{v} - \vec{u}$ and $\vec{w} - \vec{u}$ are collinear

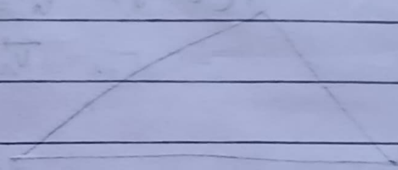
$\therefore (\vec{v} - \vec{u}) \times (\vec{w} - \vec{u}) = \vec{0}$

i.e. $\vec{u} \times \vec{v} + \vec{v} \times \vec{w} + \vec{w} \times \vec{u} = \vec{0}$



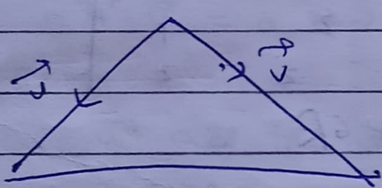
Area of Δ UVW





Area of triangle $= \frac{1}{2}\left|(\vec{v}-\vec{u})\wedge(\vec{w}-\vec{u})\right|$

$= \frac{1}{2}\left|\vec{u}\times\vec{v}+\vec{v}\times\vec{w}+\vec{w}\times\vec{u}\right|$



$\Delta = \frac{1}{2}\left|\vec{u}\times\vec{v}\right|$

$\vec{u}\times\vec{v}+\vec{v}\times\vec{w}+\vec{w}\times\vec{u}=\vec{0}$

$\Leftrightarrow (\vec{v}-\vec{u})\times(\vec{w}-\vec{u})=\vec{0}$

$\Leftrightarrow (\vec{u}-\vec{v})\times(\vec{w}-\vec{v})=\vec{0}$

$\Leftrightarrow (\vec{u}-\vec{w})\times(\vec{v}-\vec{w})=\vec{0}$

$\Leftrightarrow$ points with pv $\vec{u}, \vec{v}, \vec{w}$ are collinear

$\Leftrightarrow$ $\vec{v}-\vec{u}$ & $\vec{w}-\vec{u}$ are collinear

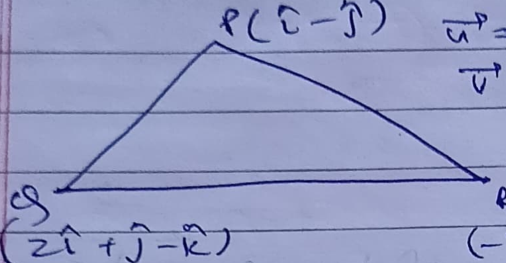
$\cancel{\Leftrightarrow}$ $\vec{u}, \vec{v}, \vec{w}$ are ~~non~~-collinear

$(\vec{u}-\vec{v})\times(\vec{u}+\vec{v}) = \vec{u}\times\vec{u}+\vec{u}\times\vec{v}-\vec{v}\times\vec{u}-\vec{v}\times\vec{v}$

$= \vec{0}+\vec{u}\times\vec{v}-\vec{v}\times\vec{u}+\vec{0}$

$= 2(\vec{u}\times\vec{v})$

Q Find a unit vector $\perp r$ to the plane of $P(1,-1,0)$, $Q(2,1,-1)$, $R(-1,1,2)$



$\vec{u} = \vec{PQ} = (2\hat{i}+\hat{j}-\hat{k}) - (\hat{i}-\hat{j}) = \hat{i}+2\hat{j}-\hat{k}$

$\vec{v} = \vec{PR} = (-\hat{i}+\hat{j}+2\hat{k}) - (\hat{i}-\hat{j})$
$= -2\hat{i}+2\hat{j}+2\hat{k}$

vector normal to $\vec{u}$ and $\vec{v}$ $= \vec{u}\times\vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix}$$

$$= 6\hat{i} + 6\hat{k}$$

Unit vector normal to $\Delta PQR = \pm\frac{\hat{i}+\hat{k}}{\sqrt{2}}$

Any vector normal to $\Delta PQR = \lambda(\hat{i}+\hat{k})$

$$t(6\hat{i}+6\hat{k}) = 6t(\hat{i}+\hat{k}) = \lambda(\hat{i}+\hat{k})$$

#

Q a) $\vec{u} = \vec{v} \Rightarrow |\vec{u}| = |\vec{v}|$
$\vec{u} = \vec{v} \Rightarrow \hat{u} = \hat{v}$

$|\vec{u}| = |\vec{v}| \not\Rightarrow \vec{u} = \vec{v}$
$\hat{u} = \hat{v} \not\Rightarrow \vec{u} = \vec{v}$

for non-zero vector $\vec{u}$ and $\vec{v}$ $\boxed{\vec{u} = \vec{v} \Leftrightarrow |\vec{u}| = |\vec{v}| \text{ AND } \hat{u} = \hat{v}}$

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6. $\vec{u} = \vec{v} \Rightarrow \vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w}$ (Taking dot product with $\vec{w}$)

$\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w} \not\Rightarrow \vec{u} = \vec{v}$

$\vec{u} = \vec{v}$

taking dot product with $\vec{w}$

$\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w}$

$\vec{u} = \vec{v} \Rightarrow \vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w}$

$\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w} \Leftrightarrow \vec{u}\cdot\vec{w} - \vec{v}\cdot\vec{w} = 0$

$\Leftrightarrow (\vec{u} - \vec{v})\cdot\vec{w} = 0$

7. $\vec{u}\cdot\vec{v} = 0 \Leftrightarrow \vec{u} = \vec{0}$ OR $\vec{v} = \vec{0}$

OR $\vec{u} \perp \vec{v}$

$\vec{u}$ ad $\vec{v}$ are non zero vectors

$\vec{u}\cdot\vec{v} = 0 \Leftrightarrow \vec{u} \perp \vec{v}$

8. $\vec{u} \times \vec{v} = \vec{0} \Rightarrow \vec{u} = \vec{0}$ OR $\vec{v} = \vec{0}$

OR $\vec{u}$ and $\vec{v}$ are collinear

$\vec{u} \times \vec{v} = \vec{0} \Leftrightarrow \vec{u}$ ad $\vec{v}$ are collinear

$\vec{u}\cdot\vec{v} = \vec{0} \Leftrightarrow \vec{u}$ ad $\vec{v}$ are orthogonal

$\vec{0}$ is orthogonal to any vector

$\vec{0}$ is collinear to any vector.

9. $\vec{u}\cdot\vec{w} = \vec{v}\cdot\vec{w}$

$\Leftrightarrow \vec{u}\cdot\vec{w} - \vec{v}\cdot\vec{w} = 0$

$\Leftrightarrow (\vec{u} - \vec{v})\cdot\vec{w} = 0$

$\Leftrightarrow \vec{u} - \vec{v}$ and $\vec{w}$ are orthogonal

(Not necessarily imply that $\vec{u} - \vec{v} = \vec{0}$
i.e. $\vec{u} = \vec{v}$)

(even if $\vec{w} \neq \vec{0}$)

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10) $\vec{u} = \vec{v} \Rightarrow \vec{u} \times \vec{w} = \vec{v} \times \vec{w}$

(taking cross product with $\vec{w}$)

$\vec{u} \times \vec{w} = \vec{v} \times \vec{w}$

$\Leftrightarrow (\vec{u} - \vec{v}) \times \vec{w} = 0$

$\Leftrightarrow \vec{u} - \vec{v}$ and $\vec{w}$ are collinear

[Does not necessarily mean $\vec{u} - \vec{v} = 0$ i.e. $\vec{u} = \vec{v}$ even if $\vec{w} \neq \vec{0}$]

$\Leftrightarrow \vec{u} - \vec{v} = t\vec{w}, \quad t \in R$

JEE 2011

5) Let $\vec{a} = -\hat{i} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j}$, $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$, then the value of $\vec{r} \cdot \vec{b}$ is ____

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$0 = \vec{r} \cdot \vec{a} = -x - z = 0$

$x = -z \qquad x + z = 0$

$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{vmatrix}$$

$\Leftrightarrow -z\hat{i} - z\hat{j} + (x+y)\hat{k} = -3\hat{i} - 3\hat{j} + 3\hat{k}$

Comparing $\hat{i}, \hat{j}, \hat{k}$

$z = 3, \qquad x + y = 3$

$x = -3$

$y = 6$

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$\vec{r}\cdot\vec{b} = \cancel{\quad} -x+y = -(-3)+6 = 9$

Alt.

$$\vec{r}\times\vec{b} = \vec{c}\times\vec{b}$$
$$\Rightarrow (\vec{r}-\vec{c})\times\vec{b} = 0$$
$$\Rightarrow \vec{r}-\vec{c} = t\vec{b}$$
$$\Rightarrow \vec{r} = \vec{c}+t\vec{b}$$
$$\Rightarrow \vec{r}\cdot\vec{a} = \vec{c}\cdot\vec{a} + t\vec{b}\cdot\vec{a} = 0$$
$$\Rightarrow t = -\frac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}} = -\left(\frac{-4}{1}\right) = 4$$

$$\vec{r} = \vec{c}+4\vec{b}$$
$$\vec{r}\cdot\vec{b} = \vec{c}\cdot\vec{b}+4\vec{b}\cdot\vec{b} = \vec{c}\cdot\vec{b}+4|\vec{b}|^2.$$
$$= 1+4\cdot 2 = 9$$

JEE 2004

Q. $\vec{a},\vec{b},\vec{c},\vec{d}$: 4 distinct vectors satisfying
$$\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$$
$$\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$$
prove that $\vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{d} \neq \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{d}$

$$\left(\begin{array}{l} \vec{u}=\vec{v} \Leftrightarrow \vec{u}+\vec{w} = \vec{v}+\vec{w} \\ \vec{u}=\vec{v} \Leftrightarrow \vec{u}-\vec{w} = \vec{v}-\vec{w} \end{array}\right)$$

$$\vec{a}\times\vec{b} = \vec{c}\times\vec{d} \quad -(1)$$
$$\vec{a}\times\vec{c} = \vec{b}\times\vec{d} \quad -(2)$$

$$\vec{a}\times\vec{b} - \vec{a}\times\vec{c} = \vec{c}\times\vec{d} - \vec{b}\times\vec{d}$$
$$\Leftrightarrow \vec{a}\times(\vec{b}-\vec{c}) = (\vec{c}-\vec{b})\times\vec{d}$$
$$\Leftrightarrow \vec{a}(\vec{b}-\vec{c}) = -(\vec{b}-\vec{c})\times\vec{d}$$
$$\Leftrightarrow \vec{a}(\vec{b}-\vec{c}) = \vec{d}\times(\vec{b}-\vec{c})$$

$$\Leftrightarrow (\vec{a}-\vec{d})\times(\vec{b}-\vec{c}) = \vec{0}$$ | $\vec{a}-\vec{d}\neq\vec{0}$, $\vec{b}-\vec{d}\neq\vec{0}$

$\Leftrightarrow (\vec{a}-\vec{d})$ and $(\vec{b}-\vec{c})$ are collinear (also non-zero)

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$\Leftrightarrow$ $(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c}) = |\vec{a}-\vec{d}|\,|\vec{b}-\vec{c}|\,(\pm 1)$

$= \pm\, |\vec{a}-\vec{d}|\,|\vec{b}-\vec{c}|$

$\neq 0$

$\Rightarrow$ $\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} - \vec{d}\cdot\vec{b} + \vec{d}\cdot\vec{c} \neq 0$

$\Rightarrow$ $\vec{a}\cdot\vec{b} + \vec{c}\cdot\vec{d} \neq \vec{b}\cdot\vec{d} + \vec{a}\cdot\vec{c}$

JEE 1998 For any two vectors

1. $(\vec{u}\cdot\vec{v})^2 + |\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2$
2. $(1+|\vec{u}|^2)(1+|\vec{v}|^2) = (1-\vec{u}\cdot\vec{v})^2 + |\vec{u}+\vec{v}+\vec{u}\times\vec{v}|^2$

RHS

$= (1-\vec{u}\cdot\vec{v})^2 + |\vec{u}+\vec{v}+\vec{u}\times\vec{v}|^2$

$= (1-\vec{u}\cdot\vec{v})^2 + |\vec{u}+\vec{v}|^2 + |\vec{u}\times\vec{v}|^2 + 2\left((\vec{u}+\vec{v})\cdot(\vec{u}\times\vec{v})\right)$

$= (1-\vec{u}\cdot\vec{v})^2 + |\vec{u}|^2 + |\vec{v}|^2 + 2(\vec{u}\cdot\vec{v}) + |\vec{u}\times\vec{v}|^2 + \vec{u}\cdot(\vec{u}\times\vec{v}) + \vec{v}\cdot(\vec{u}\times\vec{v})$

$= 1 + (\vec{u}\cdot\vec{v})^2 - \cancel{2\vec{u}\cdot\vec{v}} + |\vec{u}|^2 + |\vec{v}|^2 + \cancel{2\vec{u}\cdot\vec{v}} + |\vec{u}\times\vec{v}|^2$

$= 1 + |\vec{u}|^2 + |\vec{v}|^2 + \left((\vec{u}\cdot\vec{v})^2 + |\vec{u}\times\vec{v}|^2\right)$

$= 1 + |\vec{u}|^2 + |\vec{v}|^2 + |\vec{u}|^2|\vec{v}|^2$

$= (1+|\vec{u}|^2)(1+|\vec{v}|^2) = \text{LHS}$