

Solution We have

$$\begin{aligned} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} &= 1 + \frac{\left(-\frac{1}{2}\right)}{1} \left(\frac{-x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \cdot 2} \left(\frac{-x}{2}\right)^2 + \dots \\ &= 1 + \frac{x}{4} + \frac{3x^2}{32} + \dots \end{aligned}$$

A.1.3 Infinite Geometric Series

From Chapter 9, Section 9.5, a sequence $a_1, a_2, a_3, \dots, a_n$ is called G.P., if $\frac{a_{k+1}}{a_k} = r$ (constant) for $k = 1, 2, 3, \dots, n-1$. Particularly, if we take $a_1 = a$, then the resulting sequence $a, ar, ar^2, \dots, ar^{n-1}$ is taken as the standard form of G.P., where a is first term and r , the common ratio of G.P.

Earlier, we have discussed the formula to find the sum of finite series $a + ar + ar^2 + \dots + ar^{n-1}$ which is given by

$$S_n = \frac{a(1-r^n)}{1-r}$$

In this section, we state the formula to find the sum of infinite geometric series $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ and illustrate the same by examples.

Let us consider the G.P. $1, \frac{2}{3}, \frac{4}{9}, \dots$

Here $a = 1, r = \frac{2}{3}$. We have

$$S_n = \frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} = 3 \left[1 - \left(\frac{2}{3}\right)^n \right] \quad \dots (1)$$

Let us study the behaviour of $\left(\frac{2}{3}\right)^n$ as n becomes larger and larger.

n	1	5	10	20
$\left(\frac{2}{3}\right)^n$	0.6667	0.1316872428	0.01734152992	0.00030072866

We observe that as n becomes larger and larger, $\left(\frac{2}{3}\right)^n$ becomes closer and closer to

zero. Mathematically, we say that as n becomes sufficiently large, $\left(\frac{2}{3}\right)^n$ becomes

sufficiently small. In other words, as $n \rightarrow \infty$, $\left(\frac{2}{3}\right)^n \rightarrow 0$. Consequently, we find that

the sum of infinitely many terms is given by $S = 3$.

Thus, for infinite geometric progression a, ar, ar^2, \dots , if numerical value of common ratio r is less than 1, then

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

In this case, $r^n \rightarrow 0$ as $n \rightarrow \infty$ since $|r| < 1$ and then $\frac{ar^n}{1-r} \rightarrow 0$. Therefore,

$$S_n \rightarrow \frac{a}{1-r} \text{ as } n \rightarrow \infty.$$

Symbolically, sum to infinity of infinite geometric series is denoted by S . Thus,

we have
$$S = \frac{a}{1-r}$$

For example

$$(i) \quad 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{1}{1 - \frac{1}{2}} = 2$$

$$(ii) \quad 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

Example 2 Find the sum to infinity of the G.P. ;

$$\frac{-5}{4}, \frac{5}{16}, \frac{-5}{64}, \dots$$

Solution Here $a = \frac{-5}{4}$ and $r = -\frac{1}{4}$. Also $|r| < 1$.

$$\text{Hence, the sum to infinity is } \frac{\frac{-5}{4}}{1 + \frac{1}{4}} = \frac{\frac{-5}{4}}{\frac{5}{4}} = -1.$$

A.1.4 Exponential Series

Leonhard Euler (1707 – 1783), the great Swiss mathematician introduced the number e in his calculus text in 1748. The number e is useful in calculus as π in the study of the circle.

Consider the following infinite series of numbers

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots \quad \dots (1)$$

The sum of the series given in (1) is denoted by the number e .

Let us estimate the value of the number e .

Since every term of the series (1) is positive, it is clear that its sum is also positive.

Consider the two sums

$$\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots + \frac{1}{n!} + \dots \quad \dots (2)$$

and $\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots + \frac{1}{2^{n-1}} + \dots \quad \dots (3)$

Observe that

$$\frac{1}{3!} = \frac{1}{6} \text{ and } \frac{1}{2^2} = \frac{1}{4}, \text{ which gives } \frac{1}{3!} < \frac{1}{2^2}$$

$$\frac{1}{4!} = \frac{1}{24} \text{ and } \frac{1}{2^3} = \frac{1}{8}, \text{ which gives } \frac{1}{4!} < \frac{1}{2^3}$$

$$\frac{1}{5!} = \frac{1}{120} \text{ and } \frac{1}{2^4} = \frac{1}{16}, \text{ which gives } \frac{1}{5!} < \frac{1}{2^4}.$$