$If\ f(x)\ satisfies\ f(x+y)=f(x).f(y)\ and\ f(1)=\frac{1}{2}\ \ find\ \sum_{r=1}^{\infty} f(x)$

Solution :

$we\ know\ that\ if\ \boldsymbol{f(x+y)=f(x).f(y)}\ \ then\ \boldsymbol{f(x)=\left(f(1)\right)^{x}}$

$$\sum_{r=1}^{\infty} f(x)=\boldsymbol{f(1)+f(2)+f(3)+\cdots \ldots\ldots\ldots\ldots f(\infty)}$$

$$=\ \frac{\mathbf{1}}{\mathbf{2}}^{\mathbf{1}}+\frac{\mathbf{1}}{\mathbf{2}}^{\mathbf{2}}+\frac{\mathbf{1}}{\mathbf{2}}^{\mathbf{3}}+\frac{\mathbf{1}}{\mathbf{2}}^{\mathbf{4}}+\cdots\infty$$

$\boldsymbol{which\ is\ a\ infinite\ gp\ with\ first\ term\ being\ equal\ to}\ \frac{1}{2}\ \boldsymbol{and\ common\ ratio\ equal\ to}\ \frac{1}{2}$

$$=\frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$=\mathbf{1}$$

NOTE :

1) $\boldsymbol{if\ f(x)\ satisfies\ \ f(x+y)=f(x)+f(y)\ then\ f(x)=kx\ \ \ ,k\in R}$
2) $\boldsymbol{If\ f(x) is\ a\ polynomial\ in\ x\ and\ satisfies\ f(x)+f\left(\frac{1}{x}\right)=f(x)f\left(\frac{1}{x}\right)then\ f(x)}$
$\boldsymbol{=1\pm x^{n}\ \ n\in R}$