Find the sum $\sum_{r=1}^{\infty} r^2 x^r$, $|x|<1$

Solution: -

let, $S = x + x^2 + x^3 + x^4 + \dots \infty$

so, $x + x^2 + x^3 + \dots \infty = \frac{x}{1-x}$

differentiate both sides with respect to x

$$1 + 2x + 3x^2 + \dots \infty = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{d}{dx}\left(\frac{x-1+1}{1-x}\right)$$

$$= \frac{d}{dx}\left(-1 + \frac{1}{1-x}\right)$$

$$1 + 2x + 3x^2 + \dots \infty = \frac{1}{(1-x)^2}$$

~~again differentiate both sides~~

multiply x on both sides.

$$x + 2x^2 + 3x^3 + \dots \infty = \frac{x}{(x-1)^2}$$

again differentiate both sides w.r.t x

$$1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots \infty = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{d}{dx}\left(\frac{x-1+1}{(x-1)^2}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{(x-1)} + \frac{1}{(x-1)^2}\right)$$

$$1 + 2^2x + 3^2x^3 + 4^2x^3 + \dots = \frac{-1}{(x-1)^2} - \frac{3}{(x-1)^3}$$

again multiply x both side to get the desired answer.

$x + 2^2 x^2 + 3^2 x^3 + 4^2 x^4 \ldots = \frac{-x}{(x-1)^2} - \frac{3x}{(x-1)^4}$

this kind of differentiation trick is also used in binomial theorem chapter.

Ex :- to calculate $\sum_{r=0}^{n} (r)\, {}^nC_r$

$$\sum_{r=0}^{n} (r^2)\, {}^nC_r$$