4) Consider the family of all circles whose centers lie on the straight line $y=x$. If this family of circle is represented by the differential equation $Py''+Qy'+1=0$, where P, Q are functions of x, y and y' (here $y'=\frac{dy}{dx}$ and $y''=\frac{d^2y}{dx^2}$), then which of the following statements is (are) true?

[Advanced 2015]

(a) $P=y+x$ (b) $P=y-x$ (c) $P+Q=1-x+y+y'+(y')^2$

(d) $P-Q=x+y-y'-(y')^2$

Solution: (b, c) Let the equation of circle be

$$x^2+y^2+2gx+2gy+c=0$$

$$\Rightarrow 2x+2yy'+2g+2gy'=0$$

$$\Rightarrow x+yy'+g+gy'=0 \qquad \text{—— (1)}$$

On differentiating again, we get

$$1+yy''+(y')^2+gy''=0 \Rightarrow g=-\left[\frac{1+(y')^2+yy''}{y''}\right]$$

On substituting the value of g in eqn (1), we get

$$x+yy'-\frac{1+(y')^2+yy''}{y''}-\left(\frac{1-(y')^2+yy''}{y''}\right)y'=0$$

$$\Rightarrow xy''+yy'y''-1-(y')^2-yy''-y'-(y')^3-yy'y''=0$$

$$\Rightarrow (x-y)y''-y'(1+y'+(y')^2)=1$$

$$\Rightarrow (y-x)y''+[1+y'+(y')^2]y'+1=0$$

$$\because Py''+Qy'+1=0$$

$$P=y-x,\ Q=1+y'+(y')^2$$

Hence, $P+Q=1-x+y+y'+(y')^2$