7. $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ 8. $\cot^{-1}(\sqrt{3})$ 9. $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

10. $\text{cosec}^{-1}(-\sqrt{2})$

Find the values of the following:

11. $\tan^{-1}(1) + \cos^{-1} -\frac{1}{2} + \sin^{-1} -\frac{1}{2}$ 12. $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2}$

13. If $\sin^{-1} x = y$, then

(A) $0 \le y \le \pi$ (B) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$

14. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to

(A) π (B) $-\frac{\pi}{3}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

2.3 Properties of Inverse Trigonometric Functions

In this section, we shall prove some important properties of inverse trigonometric functions. It may be mentioned here that these results are valid within the principal value branches of the corresponding inverse trigonometric functions and wherever they are defined. Some results may not be valid for all values of the domains of inverse trigonometric functions. In fact, they will be valid only for some values of x for which inverse trigonometric functions are defined. We will not go into the details of these values of x in the domain as this discussion goes beyond the scope of this text book.

Let us recall that if $y = \sin^{-1}x$, then $x = \sin y$ and if $x = \sin y$, then $y = \sin^{-1}x$. This is equivalent to

$$\sin(\sin^{-1} x) = x,\ x \in [-1, 1] \text{ and } \sin^{-1}(\sin x) = x,\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Same is true for other five inverse trigonometric functions as well. We now prove some properties of inverse trigonometric functions.

1. (i) $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1} x,\ x \ge 1 \text{ or } x \le -1$

(ii) $\cos^{-1}\frac{1}{x} = \sec^{-1}x,\ x \ge 1 \text{ or } x \le -1$

(iii) $\tan^{-1}\frac{1}{x} = \cot^{-1} x, x > 0$

To prove the first result, we put $\text{cosec}^{-1} x = y$, i.e., $x = \text{cosec } y$

Therefore $\frac{1}{x} = \sin y$

Hence $\sin^{-1}\frac{1}{x} = y$

or $\sin^{-1}\frac{1}{x} = \text{cosec}^{-1} x$

Similarly, we can prove the other parts.

2. (i) $\sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x, x \in \mathbf{R}$

(iii) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x, |x| \geq 1$

Let $\sin^{-1}(-x) = y$, i.e., $-x = \sin y$ so that $x = -\sin y$, i.e., $x = \sin(-y)$.

Hence $\sin^{-1} x = -y = -\sin^{-1}(-x)$

Therefore $\sin^{-1}(-x) = -\sin^{-1} x$

Similarly, we can prove the other parts.

3. (i) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$

(ii) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1$

(iii) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in \mathbf{R}$

Let $\cos^{-1}(-x) = y$ i.e., $-x = \cos y$ so that $x = -\cos y = \cos(\pi - y)$

Therefore $\cos^{-1} x = \pi - y = \pi - \cos^{-1}(-x)$

Hence $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly, we can prove the other parts.

4. (i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1, 1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in \mathbf{R}$

(iii) $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}, |x| \geq 1$

Let $\sin^{-1} x = y$. Then $x = \sin y = \cos\left(\frac{\pi}{2} - y\right)$

Therefore $\cos^{-1} x = \frac{\pi}{2} - y = \frac{\pi}{2} - \sin^{-1} x$

Hence $\qquad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Similarly, we can prove the other parts.

5. (i) $\tan^{-1}x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, xy < 1$

(ii) $\tan^{-1}x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, xy > -1$

(iii) $\tan^{-1}x + \tan^{-1} y = \pi + \tan^{-1} \left(\frac{x+y}{1-xy}\right), xy > 1; x, y > 0$

Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$. Then $x = \tan \theta$, $y = \tan \phi$

Now $\qquad \tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{x+y}{1-xy}$

This gives $\qquad \theta + \phi = \tan^{-1} \frac{x+y}{1-xy}$

Hence $\qquad \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

In the above result, if we replace y by $-y$, we get the second result and by replacing y by x, we get the third result as given below.

6. (i) $2\tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}, |x| \le 1$

(ii) $2\tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}, x \ge 0$

(iii) $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, -1 < x < 1$

Let $\tan^{-1} x = y$, then $x = \tan y$. Now

$$\sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} \frac{2\tan y}{1+\tan^2 y}$$

$$= \sin^{-1} (\sin 2y) = 2y = 2\tan^{-1} x$$