**17.** If A and B are events such that $P(A|B) = P(B|A)$, then

(A) $A \subset B$ but $A \neq B$ (B) $A = B$

(C) $A \cap B = \phi$ (D) $P(A) = P(B)$

## 13.3 Multiplication Theorem on Probability

Let E and F be two events associated with a sample space S. Clearly, the set $E \cap F$ denotes the event that both E and F have occurred. In other words, $E \cap F$ denotes the simultaneous occurrence of the events E and F. The event $E \cap F$ is also written as EF.

Very often we need to find the probability of the event EF. For example, in the experiment of drawing two cards one after the other, we may be interested in finding the probability of the event 'a king and a queen'. The probability of event EF is obtained by using the conditional probability as obtained below :

We know that the conditional probability of event E given that F has occurred is denoted by $P(E|F)$ and is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$$

From this result, we can write

$$P(E \cap F) = P(F) \ . \ P(E|F) \qquad \text{... (1)}$$

Also, we know that

$$P(F|E) = \frac{P(F \cap E)}{P(E)}, P(E) \neq 0$$

or

$$P(F|E) = \frac{P(E \cap F)}{P(E)} \quad (\text{since } E \cap F = F \cap E)$$

Thus,

$$P(E \cap F) = P(E).\ P(F|E) \qquad \text{.... (2)}$$

Combining (1) and (2), we find that

$$P(E \cap F) = P(E)\ P(F|E)$$
$$= P(F)\ P(E|F) \text{ provided } P(E) \neq 0 \text{ and } P(F) \neq 0.$$

The above result is known as the *multiplication rule of probability*.

Let us now take up an example.

**Example 8** An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

**Solution** Let E and F denote respectively the events that first and second ball drawn are black. We have to find $P(E \cap F)$ or $P(EF)$.

Now $\qquad P(E) = P \text{ (black ball in first draw)} = \frac{10}{15}$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. $\qquad P(F|E) = \frac{9}{14}$

By multiplication rule of probability, we have

$$P(E \cap F) = P(E)\ P(F|E)$$

$$= \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$$

***Multiplication rule of probability for more than two events*** If E, F and G are three events of sample space, we have

$$P(E \cap F \cap G) = P(E)\ P(F|E)\ P(G|(E \cap F)) = P(E)\ P(F|E)\ P(G|EF)$$

Similarly, the multiplication rule of probability can be extended for four or more events.

The following example illustrates the extension of multiplication rule of probability for three events.

**Example 9** Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and the third card drawn is an ace?

**Solution** Let K denote the event that the card drawn is king and A be the event that the card drawn is an ace. Clearly, we have to find P (KKA)

Now $\qquad P(K) = \frac{4}{52}$

Also, P (K|K) is the probability of second king with the condition that one king has already been drawn. Now there are three kings in $(52 - 1) = 51$ cards.

Therefore $\qquad P(K|K) = \frac{3}{51}$

Lastly, P(A|KK) is the probability of third drawn card to be an ace, with the condition that two kings have already been drawn. Now there are four aces in left 50 cards.

Therefore $$P(A|KK) = \frac{4}{50}$$

By multiplication law of probability, we have

$$P(KKA) = P(K) \quad P(K|K) \quad P(A|KK)$$

$$= \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$$

## 13.4 Independent Events

Consider the experiment of drawing a card from a deck of 52 playing cards, in which the elementary events are assumed to be equally likely. If E and F denote the events 'the card drawn is a spade' and 'the card drawn is an ace' respectively, then

$$P(E) = \frac{13}{52} = \frac{1}{4} \text{ and } P(F) = \frac{4}{52} = \frac{1}{13}$$

Also E and F is the event ' the card drawn is the ace of spades' so that

$$P(E \cap F) = \frac{1}{52}$$

Hence $$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{52}}{\frac{1}{13}} = \frac{1}{4}$$

Since $P(E) = \frac{1}{4} = P\ (E|F)$, we can say that the occurrence of event F has not affected the probability of occurrence of the event E.

We also have

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{1}{52}}{\frac{1}{4}} = \frac{1}{13} = P(F)$$

Again, $P(F) = \frac{1}{13} = P(F|E)$ shows that occurrence of event E has not affected the probability of occurrence of the event F.

Thus, E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other.

Such events are called *independent events*.

**Definition 2** Two events E and F are said to be independent, if

$$P(F|E) = P(F) \text{ provided } P(E) \neq 0$$

and
$$P(E|F) = P(E) \text{ provided } P(F) \neq 0$$

Thus, in this definition we need to have $P(E) \neq 0$ and $P(F) \neq 0$

Now, by the multiplication rule of probability, we have

$$P(E \cap F) = P(E) \cdot P(F|E) \quad \text{... (1)}$$

If E and F are independent, then (1) becomes

$$P(E \cap F) = P(E) \cdot P(F) \quad \text{... (2)}$$

Thus, using (2), the independence of two events is also defined as follows:

**Definition 3** Let E and F be two events associated with the same random experiment, then E and F are said to be independent if

$$P(E \cap F) = P(E) \cdot P(F)$$

***Remarks***

(i) Two events E and F are said to be dependent if they are not independent, i.e. if

$$P(E \cap F) \neq P(E) \cdot P(F)$$

(ii) Sometimes there is a confusion between independent events and mutually exclusive events. Term 'independent' is defined in terms of 'probability of events' whereas mutually exclusive is defined in term of events (subset of sample space). Moreover, mutually exclusive events never have an outcome common, but independent events, may have common outcome. Clearly, 'independent' and 'mutually exclusive' do not have the same meaning.

In other words, two independent events having nonzero probabilities of occurrence can not be mutually exclusive, and conversely, i.e. two mutually exclusive events having nonzero probabilities of occurrence can not be independent.

(iii) Two experiments are said to be independent if for every pair of events E and F, where E is associated with the first experiment and F with the second experiment, the probability of the simultaneous occurrence of the events E and F when the two experiments are performed is the product of P(E) and P(F) calculated separately on the basis of two experiments, i.e., $P(E \cap F) = P(E) \cdot P(F)$

(iv) Three events A, B and C are said to be mutually independent, if

$$P(A \cap B) = P(A)\, P(B)$$

$$P(A \cap C) = P(A)\, P(C)$$

$$P(B \cap C) = P(B)\, P(C)$$

and
$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

If at least one of the above is not true for three given events, we say that the events are not independent.

**Example 10** A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent ?

**Solution** We know that the sample space is S = {1, 2, 3, 4, 5, 6}

Now $\quad E = \{3, 6\}, F = \{2, 4, 6\}$ and $E \cap F = \{6\}$

Then $\quad P(E) = \frac{2}{6} = \frac{1}{3},\ P(F) = \frac{3}{6} = \frac{1}{2}$ and $P(E \cap F) = \frac{1}{6}$

Clearly $\quad P(E \cap F) = P(E).\ P(F)$

Hence $\quad$ E and F are independent events.

**Example 11** An unbiased die is thrown twice. Let the event A be 'odd number on the first throw' and B the event 'odd number on the second throw'. Check the independence of the events A and B.

**Solution** If all the 36 elementary events of the experiment are considered to be equally likely, we have

$$P(A) = \frac{18}{36} = \frac{1}{2} \text{ and } P(B) = \frac{18}{36} = \frac{1}{2}$$

Also $\quad P(A \cap B) = P$ (odd number on both throws)

$$= \frac{9}{36} = \frac{1}{4}$$

Now $\quad P(A)\ P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Clearly $\quad P(A \cap B) = P(A) \times P(B)$

Thus, $\quad$ A and B are independent events

**Example 12** Three coins are tossed simultaneously. Consider the event E 'three heads or three tails', F 'at least two heads' and G 'at most two heads'. Of the pairs (E,F), (E,G) and (F,G), which are independent? which are dependent?

**Solution** The sample space of the experiment is given by

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Clearly $\quad$ E = {HHH, TTT}, F= {HHH, HHT, HTH, THH} and $\quad G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also $\quad E \cap F = \{HHH\}, E \cap G = \{TTT\}, F \cap G = \{ HHT, HTH, THH\}$

Therefore $\quad P(E) = \frac{2}{8} = \frac{1}{4}, P(F) = \frac{4}{8} = \frac{1}{2}, P(G) = \frac{7}{8}$

and $\quad P(E \cap F) = \frac{1}{8}, P(E \cap G) = \frac{1}{8}, P(F \cap G) = \frac{3}{8}$

Also $\quad P(E) \,.\, P(F) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}, P(E) \cdot P(G) = \frac{1}{4} \times \frac{7}{8} = \frac{7}{32}$

and $\quad P(F) \,.\, P(G) = \frac{1}{2} \times \frac{7}{8} = \frac{7}{16}$

Thus $\quad P(E \cap F) = P(E) \,.\, P(F)$

$P(E \cap G) \neq P(E) \,.\, P(G)$

and $\quad P(F \cap G) \neq P(F) \,.\, P(G)$

Hence, the events (E and F) are independent, and the events (E and G) and (F and G) are dependent.

**Example 13** Prove that if E and F are independent events, then so are the events E and F′.

**Solution** Since E and F are independent, we have

$$P(E \cap F) = P(E) \,.\, P(F) \quad \text{....(1)}$$

From the venn diagram in Fig 13.3, it is clear that $E \cap F$ and $E \cap F'$ are mutually exclusive events and also $E = (E \cap F) \cup (E \cap F')$.

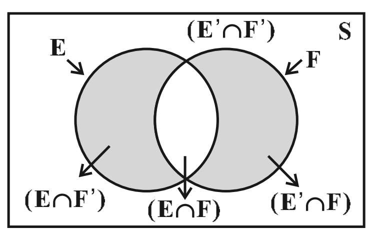

Fig 13.3

Therefore $\quad P(E) = P(E \cap F) + P(E \cap F')$

or $\quad P(E \cap F') = P(E) - P(E \cap F)$

$= P(E) - P(E) \,.\, P(F)$ (by (1))

$= P(E)\ (1-P(F))$

$= P(E).\ P(F')$

Hence, E and F′ are independent

**Note** In a similar manner, it can be shown that if the events E and F are independent, then

(a) $E'$ and F are independent,

(b) $E'$ and $F'$ are independent

**Example 14** If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by $1 - P(A')\,P(B')$

**Solution** We have

$$
\begin{aligned}
P(\text{at least one of A and B}) &= P(A \cup B)\\
&= P(A) + P(B) - P(A \cap B)\\
&= P(A) + P(B) - P(A)\,P(B)\\
&= P(A) + P(B)\,[1 - P(A)]\\
&= P(A) + P(B).\,P(A')\\
&= 1 - P(A') + P(B)\,P(A')\\
&= 1 - P(A')\,[1 - P(B)]\\
&= 1 - P(A')\,P(B')
\end{aligned}
$$

## EXERCISE 13.2

1. If $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$, find $P(A \cap B)$ if A and B are independent events.
2. Two cards are drawn at random and without replacement from a pack of 52 playing cards. Find the probability that both the cards are black.
3. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale, otherwise, it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.
4. A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die'. Check whether A and B are independent events or not.
5. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even,' and B be the event, 'the number is red'. Are A and B independent?
6. Let E and F be events with $P(E) = \frac{3}{5}$, $P(F) = \frac{3}{10}$ and $P(E \cap F) = \frac{1}{5}$. Are E and F independent?