Since the coins are fair, we can assign the probability $\frac{1}{8}$ to each sample point. Let E be the event 'at least two heads appear' and F be the event 'first coin shows tail'. Then

$$E = \{HHH, HHT, HTH, THH\}$$

and $$F = \{THH, THT, TTH, TTT\}$$

Therefore $$P(E) = P(\{HHH\}) + P(\{HHT\}) + P(\{HTH\}) + P(\{THH\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2} \text{ (Why ?)}$$

and $$P(F) = P(\{THH\}) + P(\{THT\}) + P(\{TTH\}) + P(\{TTT\})$$

$$= \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

Also $$E \cap F = \{THH\}$$

with $$P(E \cap F) = P(\{THH\}) = \frac{1}{8}$$

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E? With the information of occurrence of F, we are sure that the cases in which first coin does not result into a tail should not be considered while finding the probability of E. This information reduces our sample space from the set S to its subset F for the event E. In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favourable to the occurrence of the event F.

Now, the sample point of F which is favourable to event E is THH.

Thus, Probability of E considering F as the sample space = $\frac{1}{4}$,

or Probability of E given that the event F has occurred = $\frac{1}{4}$

This probability of the event E is called the *conditional probability of E given that F has already occurred*, and is denoted by P (E|F).

Thus $$P(E|F) = \frac{1}{4}$$

Note that the elements of F which favour the event E are the common elements of E and F, i.e. the sample points of $E \cap F$.

Thus, we can also write the conditional probability of E given that F has occurred as

$$P(E|F) = \frac{\text{Number of elementary events favourable to } E \cap F}{\text{Number of elementary events which are favourable to } F}$$

$$= \frac{n(E \cap F)}{n(F)}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that P(E|F) can also be written as

$$P(E|F) = \frac{\dfrac{n(E \cap F)}{n(S)}}{\dfrac{n(F)}{n(S)}} = \frac{P(E \cap F)}{P(F)} \qquad \text{... (1)}$$

Note that (1) is valid only when $P(F) \neq 0$ i.e., $F \neq \phi$ (Why?)

Thus, we can define the conditional probability as follows :

Definition 1 If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e. P (E|F) is given by

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \text{ provided } P(F) \neq 0$$

13.2.1 *Properties of conditional probability*

Let E and F be events of a sample space S of an experiment, then we have

Property 1 $P(S|F) = P(F|F) = 1$

We know that

$$P(S|F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Also

$$P(F|F) = \frac{P(F \cap F)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

Thus

$$P(S|F) = P(F|F) = 1$$

Property 2 *If* A *and* B *are any two events of a sample space* S *and* F *is an event of* S *such that* $P(F) \neq 0$*, then*

$$P((A \cup B)|F) = P(A|F) + P(B|F) - P((A \cap B)|F)$$

In particular, if A *and* B *are disjoint events, then*

$$P((A \cup B)|F) = P(A|F) + P(B|F)$$

We have

$$P((A \cup B)|F) = \frac{P[(A \cup B) \cap F]}{P(F)}$$

$$= \frac{P[(A \cap F) \cup (B \cap F)]}{P(F)}$$

(by distributive law of union of sets over intersection)

$$= \frac{P(A \cap F) + P(B \cap F) - P(A \cap B \cap F)}{P(F)}$$

$$= \frac{P(A \cap F)}{P(F)} + \frac{P(B \cap F)}{P(F)} - \frac{P[(A \cap B) \cap F]}{P(F)}$$

$$= P(A|F) + P(B|F) - P((A \cap B)|F)$$

When A and B are disjoint events, then

$$P((A \cap B)|F) = 0$$

$$\Rightarrow \quad P((A \cup B)|F) = P(A|F) + P(B|F)$$

Property 3 $P(E'|F) = 1 - P(E|F)$

From Property 1, we know that $P(S|F) = 1$

$\Rightarrow \quad P(E \cup E'|F) = 1 \quad$ since $S = E \cup E'$

$\Rightarrow \quad P(E|F) + P(E'|F) = 1 \quad$ since E and E′ are disjoint events

Thus, $\quad P(E'|F) = 1 - P(E|F)$

Let us now take up some examples.

Example 1 If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, evaluate $P(A|B)$.

Solution We have $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{4}{13}}{\frac{9}{13}} = \dfrac{4}{9}$

Example 2 A family has two children. What is the probability that both the children are boys given that at least one of them is a boy ?