Q5. Sometimes a radioactive nucleus decays into a nucleus which itself is radioactive. An example is:

$$^{38}\text{Sulphur} \xrightarrow[=2.48\,h]{\text{half-life}} {}^{38}Cl \xrightarrow[=0.62\,h]{\text{half-life}} {}^{38}Ar\ (\text{stable})$$

Assume that we start with 1000 ^{38}S nuclei at time $t=0$. The number of ^{38}Cl is of count zero at $t=0$ and will again be zero at $t=\infty$. At what value of t, would the number of counts be a maximum.

$$S \xrightarrow{\lambda_1} Cl \xrightarrow{\lambda_2} Ar$$

At any time t,

$\frac{dN_{Cl}}{dt}$ for $Cl = \lambda_1 N_S - \lambda_2 N_{Cl}$. When N_{Cl} is max $\Rightarrow \frac{dN_{Cl}}{dt} = 0$

$$N_S = N_0 e^{-\lambda_1 t}$$

$$\frac{dN}{dt} = \lambda_1 N_0 e^{-\lambda_1 t} - \lambda_2 N$$

$$\frac{dN}{dt} + \lambda_2 N = \lambda_1 N_0 e^{-\lambda_1 t}$$

$$e^{\lambda_2 t}\frac{dN}{dt} + \lambda_2 e^{\lambda_2 t} N = \lambda_1 N_0 e^{(\lambda_2-\lambda_1)t}$$ [Integrating factor $= e^{\lambda_2 t}$]

$$\frac{d}{dt}\left(Ne^{\lambda_2 t}\right) = \lambda_1 N_0 e^{(\lambda_2-\lambda_1)t}$$

$$Ne^{\lambda_2 t} = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\left[e^{(\lambda_2-\lambda_1)t} - 1\right]$$

$$N_{Cl} = \frac{\lambda_1 N_0}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$\frac{dN_{Cl}}{dt} = 0 \Rightarrow \lambda_1 N_S = \lambda_2 N_{Cl}$$

$$\Rightarrow \quad \lambda_1 N_0 e^{-\lambda_1 t} = \frac{\lambda_2 \lambda_1 N_0}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$(\lambda_2 - \lambda_1) e^{-\lambda_1 t} = \lambda_2 \left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$-\lambda_1 e^{-\lambda_1 t} = -\lambda_2 e^{-\lambda_2 t}$$

$$e^{(\lambda_2 - \lambda_1)t} = \frac{\lambda_2}{\lambda_1}$$

$$(\lambda_2 - \lambda_1)\, t = \ln\left(\frac{\lambda_2}{\lambda_1}\right)$$

$$t = \frac{1}{(\lambda_2 - \lambda_1)} \ln\left(\frac{\lambda_2}{\lambda_1}\right)$$

$$\lambda_1 = \frac{\ln 2}{2.48} \ \text{hr}^{-1}$$

$$\lambda_2 = \frac{\ln 2}{0.62} \ \text{hr}^{-1}$$

$$\Rightarrow \quad t = \frac{1}{\ln 2\left(\frac{1}{0.62} - \frac{1}{2.48}\right)} \ln\left(\frac{2.48}{0.62}\right) = \frac{\ln 4 \times 0.62 \times 2.48}{\ln 2\,(2.48 - 0.62)} = 1.65 \ \text{hrs}$$