Question 11: If the least and the largest real values of α, for which the equation $z + \alpha|z-1| + 2i = 0$ ($z \in C$ and $i = \sqrt{(-1)}$) has a solution, are p and q respectively; then $4(p^2 + q^2)$ is equal to

Solution:

Let $z = x+iy$

Given $z + \alpha|z-1| + 2i = 0$

$\Rightarrow x + iy + \alpha\sqrt{((x-1)^2 + y^2)} + 2i = 0$

Comparing imaginary coefficients

$y + 2 = 0$

$y = -2$

Comparing real coefficients

$x + \alpha\sqrt{((x-1)^2+y^2)} = 0$

$x^2 = \alpha^2(x^2 - 2x + 1 + 4)$

$\alpha^2 = x^2/(x^2 - 2x + 5)$

$\alpha^2(x^2 - 2x + 5) = x^2$

$\Rightarrow x^2(\alpha^2 - 1) - 2x\alpha^2 + 5\alpha^2 = 0$

Since $x \in R$, $D \geq 0$

$\Rightarrow 4\alpha^4 - 4(\alpha^2 - 1)5\alpha^2 \geq 0$

$\Rightarrow \alpha^2[4\alpha^2 - 20\alpha^2 + 20] \geq 0$

$\Rightarrow \alpha^2[-16\alpha^2 + 20] \geq 0$

$\Rightarrow \alpha^2[\alpha^2 - 5/4] \leq 0$

$\Rightarrow \alpha^2 \in [0, 5/4]$

$\Rightarrow \alpha \in [-\sqrt{5}/2, \sqrt{5}/2]$

then $4[p^2 + q^2] = 4[5/4 + 5/4]$

$= 10$