Que 10: For every twice differentiable function $f: R \to [-2,2]$ with $\left(f(0)\right)^2 + \left(f'(0)\right)^2 = 85$ which of the following statement(s) is(are) TRUE?

(1) There exist $r, s \in R$, where $r < s$, such that f is one-one on the open interval (r, s)
(2) There exists $x_0 \in (-4,0)$ such that $|f'(x_0)| \leq 1$
(3) $\lim_{x \to \infty} f(x) = 1$

(4) There exists $\alpha \in (-4,4)$ such that $f(\alpha) - f''(\alpha) = 0$ and $f(\alpha) \neq 0$

Ans 10:

$f(x)$ can't be constant throughout the domain. Hence we can find$x \in (r, s)$ such that $f(x)$ is one-one

Option (1) is true.

Option (2): $|f'(x_0)| = \left|\frac{f(0)-f(-4)}{4}\right| \leq 1\ (LMVT)$

Option (3): $f(x) = \sin(\sqrt{85}x)$ satisfies given condition

But $\lim_{x\to\infty} \sin\left(\sqrt{85}\right)$ does not exist ⇒incorrect

Option (4): $g(x) = f^2(x) + \left(f'(x)\right)^2$

$$|f'(x_1)| \leq 1 \quad (by\ LMVT)$$
$$|f(x_1)| \leq 2 \quad (given)$$
$$\Rightarrow g(x_1) \leq 5\ \exists x_1 \in (-4,0)$$

Similarly, $g(x_2) \leq 5 \quad \exists x_2 \in (0,4)$
$g(0) = 85 \Rightarrow g(x)$ has minima in (x_1, x_2) say at α.

$$g'(\alpha) = 0\ \&g(\alpha) \geq 85$$
$$2f'(\alpha)\left(f(\alpha) + f''(\alpha)\right) = 0$$

If $f'(\alpha) = 0 \Rightarrow g(\alpha) = f^2(\alpha) \geq not\ possible$

$$\Rightarrow f(\alpha) + f''(\alpha) = 0 \quad \exists \alpha \in (x_1, x_2) \in (-4,4)$$

Option (4) is correct.