Example 41 An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at $(3, 7)$, wants to shoot down the helicopter when it is nearest to him. Find the nearest distance.

Solution For each value of x, the helicopter's position is at point $(x, x^2 + 7)$. Therefore, the distance between the helicopter and the soldier placed at $(3,7)$ is

$$\sqrt{(x-3)^2 + (x^2+7-7)^2}, \text{ i.e., } \sqrt{(x-3)^2 + x^4}.$$

Let $$f(x) = (x-3)^2 + x^4$$

or $$f'(x) = 2(x-3) + 4x^3 = 2(x-1)(2x^2 + 2x + 3)$$

Thus, $f'(x) = 0$ gives $x = 1$ or $2x^2 + 2x + 3 = 0$ for which there are no real roots. Also, there are no end points of the interval to be added to the set for which f' is zero, i.e., there is only one point, namely, $x = 1$. The value of f at this point is given by $f(1) = (1-3)^2 + (1)^4 = 5$. Thus, the distance between the solider and the helicopter is $\sqrt{f(1)} = \sqrt{5}$.

Note that $\sqrt{5}$ is either a maximum value or a minimum value. Since

$$\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5},$$

it follows that $\sqrt{5}$ is the minimum value of $\sqrt{f(x)}$. Hence, $\sqrt{5}$ is the minimum distance between the soldier and the helicopter.