Q2 : Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

Answer :

The given function is $f(x) = \frac{\log x}{x}$.

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

Now, $f'(x) = 0$

$\Rightarrow 1 - \log x = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log x = \log e$

$\Rightarrow x = e$

$$\text{Now, } f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$$

$$= \frac{-x - 2x(1 - \log x)}{x^4}$$

$$= \frac{-3 + 2\log x}{x^3}$$

$$\text{Now, } f''(e) = \frac{-3 + 2\log e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3} < 0$$

Therefore, by second derivative test, f is the maximum at $x = e$.