If $\vec{b}$ and $\vec{c}$ are any two non-collinear unit vectors and $\vec{a}$ is any vector, then

$$(\vec{a} \cdot \vec{b})\vec{b} + (\vec{a} \cdot \vec{c})\vec{c} + \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|^2}(\vec{b} \times \vec{c}) = \ldots\ldots$$

(1996, 2M)

Let $\hat{\mathbf{i}}$ be a unit vector in the direction of $\vec{\mathbf{b}}$, $\hat{\mathbf{j}}$ in the direction of $\vec{\mathbf{c}}$. Note that $\vec{\mathbf{c}} = \hat{\mathbf{j}}$

and $$(\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = |\vec{\mathbf{b}}||\vec{\mathbf{c}}| \sin \alpha \, \hat{\mathbf{k}} = \sin \alpha \, \hat{\mathbf{k}}$$

where, $\hat{\mathbf{k}}$ is a unit vector perpendicular to $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$.

$$\Rightarrow \quad |\vec{\mathbf{b}} \times \vec{\mathbf{c}}| = \sin \alpha \quad \Rightarrow \quad \hat{\mathbf{k}} = \frac{\vec{\mathbf{b}} \times \vec{\mathbf{c}}}{|\vec{\mathbf{b}} \times \vec{\mathbf{c}}|}$$

Let $$\vec{\mathbf{a}} = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}$$

Now, $$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{a}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{i}} \cdot (a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}) = a_1$$

and $$\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot (a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}) = a_2$$

and $$\vec{\mathbf{a}} \cdot \frac{\vec{\mathbf{b}} \times \vec{\mathbf{c}}}{|\vec{\mathbf{b}} \times \vec{\mathbf{c}}|} = \vec{\mathbf{a}} \cdot \hat{\mathbf{k}} = a_3$$

$$\therefore (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \vec{\mathbf{b}} + (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) \vec{\mathbf{c}} + \frac{\vec{\mathbf{a}} \cdot (\vec{\mathbf{b}} \times \vec{\mathbf{c}})}{|\vec{\mathbf{b}} \times \vec{\mathbf{c}}|^2} (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$$

$$= a_1 \vec{\mathbf{b}} + a_2 \vec{\mathbf{c}} + a_3 \frac{(\vec{\mathbf{b}} \times \vec{\mathbf{c}})}{|\vec{\mathbf{b}} \times \vec{\mathbf{c}}|} = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}} = \vec{\mathbf{a}}$$