Stem for Question Nos. 5 and 6

Consider the lines L_1 and L_2 defined by

$L_1 : x\sqrt{2} + y - 1 = 0$ and $L_2 : x\sqrt{2} - y + 1 = 0$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ_2. The line $y = 2x + 1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the **square** of the distance between R' and S'.

Q5. The value of λ^2 is ______. **[Adv. 2021]**

Q6. The value of D is ______. **[Adv. 2021]**

Sol 5,6 Let locus point $P(x, y)$.

$\therefore$ According to question,

$$\left|\frac{\sqrt{2}x+y-1}{\sqrt{3}}\right|\left|\frac{\sqrt{2}x-y+1}{\sqrt{3}}\right|=\lambda^2$$

$$\Rightarrow \left|\frac{2x^2-(y-1)^2}{3}\right|=\lambda^2$$

So, $C : |2x^2-(y-1)^2| = 3\lambda^2$

Let the line $y = 2x + 1$ meets C at two points $R(x_1, y_1)$ and $S(x_2, y_2)$

$\Rightarrow y_1 = 2x_1 + 1$ and $y_2 = 2x_2 + 1$

$\Rightarrow (y_1 - y_2) = 2(x_1 - x_2)$

$\therefore \quad RS = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$RS = \sqrt{5(x_1 - x_2)^2} = \sqrt{5}\,|x_1 - x_2|$

On solving equations curve C and line $y = 2x + 1$, we get

$|2x^2 - (2x)^2| = 3\lambda^2 \quad \Rightarrow \quad x^2 = \frac{3\lambda^2}{2}$

$\therefore \quad RS = \sqrt{5}\left|\frac{2\sqrt{3}\lambda}{\sqrt{2}}\right| = \sqrt{30}\,\lambda = \sqrt{270}$

$\Rightarrow \quad 30\lambda^2 = 270 \Rightarrow \lambda^2 = 9$

$$T \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Since, $x_1 + x_2 = 0 \qquad \Rightarrow \; y_1 - y_2 = 0$

So, $T = (0, 1)$

Equation of $R'S'$:

$$(y-1) = -\frac{1}{2}(x-0) \Rightarrow x + 2y = 2$$

Let $R'(a_1, b_1)$ and $S'(a_2, b_2)$

$\therefore \quad D = (a_1 - a_2)^2 + (b_1 - b_2)^2 = 5\,(b_1 - b_2)^2$

On solving $x + 2y = 2$ and $|2x^2 - (y-1)^2| = 3\lambda^2$, we get

$$\Rightarrow \; |8\,(y-1)^2 - (y-1)^2| = 3\lambda^2$$

$$\Rightarrow \; (y-1)^2 = \left(\frac{\sqrt{3}\lambda}{\sqrt{7}}\right)^2$$

$$y - 1 = \pm\frac{\sqrt{3}\lambda}{\sqrt{7}}$$

$$\Rightarrow \; b_1 = 1 + \frac{\sqrt{3}\lambda}{\sqrt{7}}, b_2 = 1 - \frac{\sqrt{3}\lambda}{\sqrt{7}}$$

$$D = 5\left(\frac{2\sqrt{3}\lambda}{\sqrt{7}}\right)^2 = \frac{5 \times 4 \times 3\lambda^2}{7} = \frac{5 \times 4 \times 27}{7} = 77.14$$