Question

A pair of stars rotates about a common centre of mass. One of the stars has a mass M and the other m. Their centres are a distance d apart, d being large compared to the size of either star. Derive an expression for the period of revolution of the stars about their common centre of mass. Compare their angular momenta and kinetic energies.

Solution

The centre of mass of the system divides the distance between the stars in the

inverse ratio of their masses. If d_1 and d_2 are the distance force for their circular motion

$$d_1 = \frac{d}{M+m} \times m$$

$$\text{and } d_2 = \frac{d}{M+m} \times M \Rightarrow \frac{d_1}{d_2} = \frac{m}{M}$$

the stars will rotate in circles of radii d_1

and d_2 about their centre of mass. The same force of attraction provides the necessary centripetal force for their circular motion.

$$\therefore \frac{GmM}{d^2} = M\omega_1^2 d_1 = m\omega_2^2 d_2$$

$$\text{or } \omega_1^2 = \frac{Gm}{d^2 d_1} = \frac{Gm}{d^2} \times \frac{M+m}{d \times m}$$

$$= \frac{G(M+m)}{d^2}$$

$$\text{and } \omega_2^2 = \frac{GM}{d^2 d_2} = \frac{GM}{d^2} \times \frac{M+m}{d \times M}$$

$$= \frac{G(M+m)}{d^2}$$

$$\omega_1 = \omega_2 \sqrt{\frac{G(M+m)}{d^2}}$$

From the fact the moment of momentum is also the angular momentum

$$\frac{L_M}{L_m} = \frac{(Mv_1)d_1}{(mv_2)d_2} = \frac{M}{m} \times \frac{d_1^2}{d_2^2} \Rightarrow \frac{L_M}{L_m} = \frac{m}{M}$$

$$\Rightarrow \frac{K_M}{K_m} = \frac{\frac{1}{2}mv_1^2}{\frac{1}{2}mv_1^2} = \frac{M}{m}\frac{\omega_1^2 d_1^2}{\omega_2^2 d_2^2} = \frac{M}{m}\frac{d_1^2}{d_2^2}$$

$\omega_1 = \omega_2)$

$$\text{Therefore, } \frac{K_M}{K_m} = \frac{M}{m}\left(\frac{m}{M}\right)^2 = \frac{m}{M}$$