

(Q) 3) By using properties of determinants, show that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Solution:

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors from R_1, R_2 and R_3 , we have:

$$\Delta = abc \begin{vmatrix} a + \frac{1}{a} & b & c \\ a & b + \frac{1}{b} & c \\ a & b & c + \frac{1}{c} \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$.

$$\Delta = abc \begin{vmatrix} a + \frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying $C_1 \rightarrow aC_1$, $C_2 \rightarrow bC_2$, and $C_3 \rightarrow cC_3$, we have

$$\Delta = abc \times \frac{1}{abc} \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along R_3 , we have:

$$\Delta = -1 \begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} a^2+1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$= -1(-c^2) + (a^2+1+b^2) = 1+a^2+b^2+c^2$$