Problem 5. (Tough)

Find the greatest value of $f(x) = (\sin^{-1}x)^3 + (\cos^{-1}x)^3$, $|x| \leq 1$.

Let $\cos^{-1}x = t \Rightarrow 0 \leq t \leq \pi$.

$\therefore \sin^{-1}x = \pi/2 - t$.

So, $g(t) = t^3 + (\pi/2 - t)^3$.

$\therefore g(t) = \frac{\pi^3}{8} - \frac{3\pi^2}{4}t + \frac{3\pi}{2}t^2$.

Now, the question is reduced to finding the maximum value of a quadratic expression $g(t)$ on the interval $[0,\pi]$

$\therefore g(t) = \frac{3\pi}{2}\left[t^2 - \frac{\pi}{2}t + \pi^2/12\right]$

$= \frac{3\pi}{2}\left[\left(t - \pi/4\right)^2 + \pi^2/48\right]$.

for max value, we should set $t = \pi$.

$\therefore$ Required max value $= g(\pi)$

$= 3\pi/2\left[\left(3\pi/4\right)^2 + \pi^2/48\right]$

$= \boxed{\frac{7}{8}\pi^3}$.