EXAMPLE 2.9

Solution

(a) In the given network, C_1, C_2 and C_3 are connected in series. The effective capacitance C′ of these three capacitors is given by

$$\frac{1}{C'} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

For $C_1 = C_2 = C_3 = 10\ \mu F$, $C' = (10/3)\ \mu F$. The network has C' and C_4 connected in parallel. Thus, the equivalent capacitance C of the network is

$$C = C' + C_4 = \left(\frac{10}{3} + 10\right)\ \mu F = 13.3 \mu F$$

(b) Clearly, from the figure, the charge on each of the capacitors, C_1, C_2 and C_3 is the same, say Q. Let the charge on C_4 be Q'. Now, since the potential difference across AB is Q/C_1, across BC is Q/C_2, across CD is Q/C_3, we have

$$\frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} = 500\ V$$

Also, $Q'/C_4 = 500$ V.

This gives for the given value of the capacitances,

$$Q = 500\,V \times \frac{10}{3} \mu F = 1.7 \times 10^{-3} C \text{ and}$$

$$Q' = 500\,V \times 10 \mu F = 5.0 \times 10^{-3}\ C$$

2.15 ENERGY STORED IN A CAPACITOR

A capacitor, as we have seen above, is a system of two conductors with charge Q and $-Q$. To determine the energy stored in this configuration, consider initially two uncharged conductors 1 and 2. Imagine next a process of transferring charge from conductor 2 to conductor 1 bit by bit, so that at the end, conductor 1 gets charge Q. By charge conservation, conductor 2 has charge $-Q$ at the end (Fig 2.30).

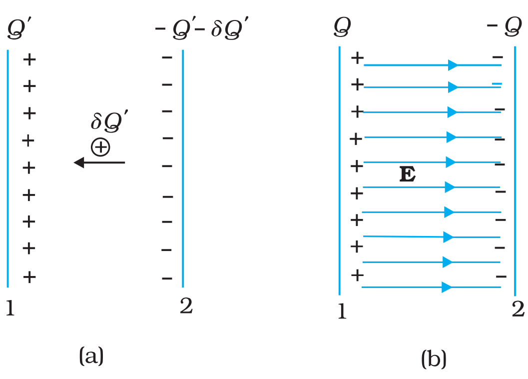

FIGURE 2.30 (a) Work done in a small step of building charge on conductor 1 from Q' to $Q' + \delta Q'$. (b) Total work done in charging the capacitor may be viewed as stored in the energy of electric field between the plates.

In transferring positive charge from conductor 2 to conductor 1, work will be done externally, since at any stage conductor 1 is at a higher potential than conductor 2. To calculate the total work done, we first calculate the work done in a small step involving transfer of an infinitesimal (i.e., vanishingly small) amount of charge. Consider the intermediate situation when the conductors 1 and 2 have charges Q' and $-Q'$ respectively. At this stage, the potential difference V' between conductors 1 to 2 is Q'/C, where C is the capacitance of the system. Next imagine that a small charge $\delta Q'$ is transferred from conductor 2 to 1. Work done in this step (δW), resulting in charge Q' on conductor 1 increasing to $Q' + \delta Q'$, is given by

$$\delta W = V' \delta Q' = \frac{Q'}{C} \delta Q' \qquad (2.68)$$

Since $\delta Q'$ can be made as small as we like, Eq. (2.68) can be written as

$$\delta W = \frac{1}{2C}[(Q' + \delta Q')^2 - Q'^2] \tag{2.69}$$

Equations (2.68) and (2.69) are identical because the term of second order in $\delta Q'$, i.e., $\delta Q'^2/2C$, is negligible, since $\delta Q'$ is arbitrarily small. The total work done (W) is the sum of the small work (δW) over the very large number of steps involved in building the charge Q' from zero to Q.

$$W = \sum_{\text{sum over all steps}} \delta W$$

$$= \sum_{\text{sum over all steps}} \frac{1}{2C}[(Q' + \delta Q')^2 - Q'^2] \tag{2.70}$$

$$= \frac{1}{2C}[\{\delta Q'^2 - 0\} + \{(2\delta Q')^2 - \delta Q'^2\} + \{(3\,\delta Q')^2 - (2\,\delta Q')^2\} + \ldots$$

$$+ \{Q^2 - (Q - \delta Q')^2\}] \tag{2.71}$$

$$= \frac{1}{2C}[Q^2 - 0] = \frac{Q^2}{2C} \tag{2.72}$$

The same result can be obtained directly from Eq. (2.68) by integration

$$W = \int_0^Q \frac{Q'}{C}\,\delta Q' = \frac{1}{C}\left.\frac{Q'^2}{2}\right|_0^Q = \frac{Q^2}{2C}$$

This is not surprising since integration is nothing but summation of a large number of small terms.

We can write the final result, Eq. (2.72) in different ways

$$W = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \tag{2.73}$$

Since electrostatic force is conservative, this work is stored in the form of potential energy of the system. For the same reason, the final result for potential energy [Eq. (2.73)] is independent of the manner in which the charge configuration of the capacitor is built up. When the capacitor discharges, this stored-up energy is released. It is possible to view the potential energy of the capacitor as 'stored' in the electric field between the plates. To see this, consider for simplicity, a parallel plate capacitor [of area A (of each plate) and separation d between the plates].

Energy stored in the capacitor

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{(A\sigma)^2}{2} \times \frac{d}{\varepsilon_0 A} \tag{2.74}$$

The surface charge density σ is related to the electric field E between the plates,

$$E = \frac{\sigma}{\varepsilon_0} \tag{2.75}$$

From Eqs. (2.74) and (2.75) , we get

Energy stored in the capacitor

$$U = (1/2)\,\varepsilon_0 E^2 \times A\,d \tag{2.76}$$

Note that Ad is the volume of the region between the plates (where electric field alone exists). If we define *energy density as energy stored per unit volume of space*, Eq (2.76) shows that

Energy density of electric field,

$$u = (1/2)\varepsilon_0 E^2 \tag{2.77}$$

Though we derived Eq. (2.77) for the case of a parallel plate capacitor, the result on energy density of an electric field is, in fact, very general and holds true for electric field due to any configuration of charges.

Example 2.10 (a) A 900 pF capacitor is charged by 100 V battery [Fig. 2.31(a)]. How much electrostatic energy is stored by the capacitor? (b) The capacitor is disconnected from the battery and connected to another 900 pF capacitor [Fig. 2.31(b)]. What is the electrostatic energy stored by the system?

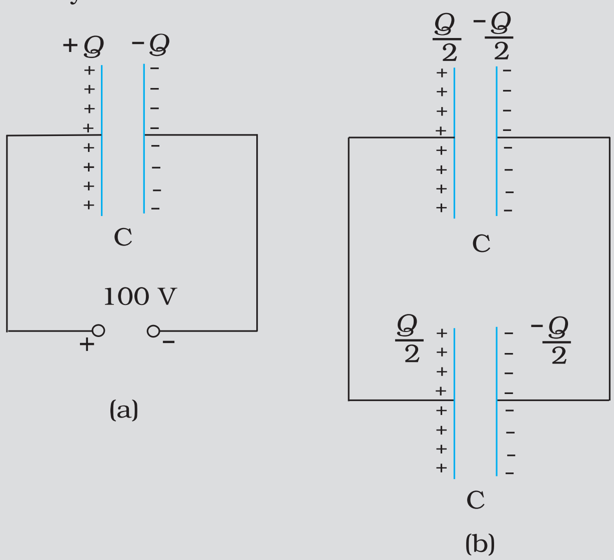

FIGURE 2.31

Solution

(a) The charge on the capacitor is

$Q = CV = 900 \times 10^{-12}\ \text{F} \times 100\ \text{V} = 9 \times 10^{-8}\ \text{C}$

The energy stored by the capacitor is

$= (1/2)\ CV^2 = (1/2)\ QV$

$= (1/2) \times 9 \times 10^{-8}\text{C} \times 100\ \text{V} = 4.5 \times 10^{-6}\ \text{J}$

(b) In the steady situation, the two capacitors have their positive plates at the same potential, and their negative plates at the same potential. Let the common potential difference be V′. The

EXAMPLE 2.10

charge on each capacitor is then $Q' = CV'$. By charge conservation, $Q' = Q/2$. This implies $V' = V/2$. The total energy of the system is $= 2 \times \frac{1}{2} Q'V' = \frac{1}{4} QV = 2.25 \times 10^{-6}$ J

Thus in going from (a) to (b), though no charge is lost; the final energy is only half the initial energy. *Where has the remaining energy gone?*

There is a transient period before the system settles to the situation (b). During this period, a transient current flows from the first capacitor to the second. Energy is lost during this time in the form of heat and electromagnetic radiation.

EXAMPLE 2.10

SUMMARY

1. Electrostatic force is a conservative force. Work done by an external force (equal and opposite to the electrostatic force) in bringing a charge q from a point R to a point P is $q(V_P–V_R)$, which is the difference in potential energy of charge q between the final and initial points.
2. Potential at a point is the work done per unit charge (by an external agency) in bringing a charge from infinity to that point. Potential at a point is arbitrary to within an additive constant, since it is the potential difference between two points which is physically significant. If potential at infinity is chosen to be zero; potential at a point with position vector **r** due to a point charge Q placed at the origin is given is given by

   $$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \frac{Q}{r}$$

3. The electrostatic potential at a point with position vector **r** due to a point dipole of dipole moment **p** placed at the origin is

   $$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_o} \frac{\mathbf{p}.\hat{\mathbf{r}}}{r^2}$$

   The result is true also for a dipole (with charges $-q$ and q separated by $2a$) for $r >> a$.
4. For a charge configuration q_1, q_2, …, q_n with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, … $\mathbf{r}_n$, the potential at a point P is given by the superposition principle

   $$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}} + \frac{q_2}{r_{2P}} + \ldots + \frac{q_n}{r_{nP}}\right)$$

   where r_{1P} is the distance between q_1 and P, as and so on.
5. An equipotential surface is a surface over which potential has a constant value. For a point charge, concentric spheres centred at a location of the charge are equipotential surfaces. The electric field **E** at a point is perpendicular to the equipotential surface through the point. **E** is in the direction of the steepest decrease of potential.