Copernicus (1473-1543) proposed a definitive model in which the planets moved in circles around a fixed central sun. His theory was discredited by the church, but notable amongst its supporters was Galileo who had to face prosecution from the state for his beliefs.

It was around the same time as Galileo, a nobleman called Tycho Brahe (1546-1601) hailing from Denmark, spent his entire lifetime recording observations of the planets with the naked eye. His compiled data were analysed later by his assistant Johannes Kepler (1571-1640). He could extract from the data three elegant laws that now go by the name of Kepler's laws. These laws were known to Newton and enabled him to make a great scientific leap in proposing his universal law of gravitation.

8.2 KEPLER'S LAWS

The three laws of Kepler can be stated as follows:

1. Law of orbits : All planets move in elliptical orbits with the Sun situated at one of the foci

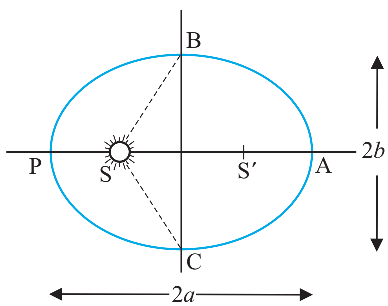


Fig. 8.1(a) *An ellipse traced out by a planet around the sun. The closest point is P and the farthest point is A, P is called the perihelion and A the aphelion. The semimajor axis is half the distance AP.*

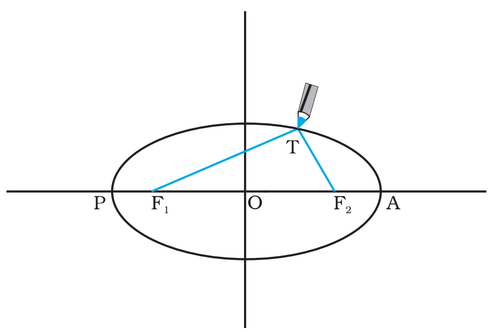


Fig. 8.1(b) *Drawing an ellipse. A string has its ends fixed at F_1 and F_2. The tip of a pencil holds the string taut and is moved around.*

of the ellipse (Fig. 8.1a). This law was a deviation from the Copernican model which allowed only circular orbits. The ellipse, of which the circle is a special case, is a closed curve which can be drawn very simply as follows.

Select two points F_1 and F_2. Take a length of a string and fix its ends at F_1 and F_2 by pins. With the tip of a pencil stretch the string taut and then draw a curve by moving the pencil keeping the string taut throughout.(Fig. 8.1(b)) The closed curve you get is called an ellipse. Clearly for any point T on the ellipse, the sum of the distances from F_1 and F_2 is a constant. F_1, F_2 are called the focii. Join the points F_1 and F_2 and extend the line to intersect the ellipse at points P and A as shown in Fig. 8.1(b). The midpoint of the line PA is the centre of the ellipse O and the length PO = AO is called the semi-major axis of the ellipse. For a circle, the two focii merge into one and the semi-major axis becomes the radius of the circle.

2. Law of areas : The line that joins any planet to the sun sweeps equal areas in equal intervals of time (Fig. 8.2). This law comes from the observations that planets appear to move slower when they are farther from the sun than when they are nearer.

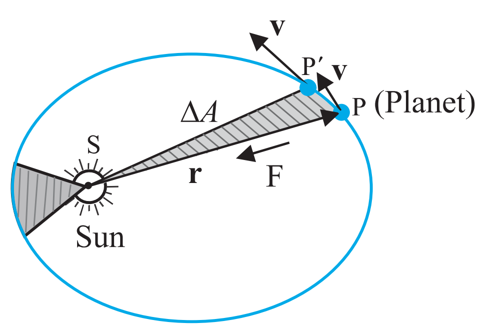


Fig. 8.2 *The planet P moves around the sun in an elliptical orbit. The shaded area is the area ΔA swept out in a small interval of time Δt.*

3. Law of periods : The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of the ellipse traced out by the planet.

Table 8.1 gives the approximate time periods of revolution of eight* planets around the sun along with values of their semi-major axes.

* *Refer to information given in the Box on Page 182*

Table 8.1 Data from measurement of planetary motions given below confirm Kepler's Law of Periods

(a ≡ Semi-major axis in units of 10^{10} m.
T ≡ Time period of revolution of the planet in years(y).
Q ≡ The quotient (T^2/a^3) in units of 10^{-34} y^2 m^{-3}.)

Planet	a	T	Q
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84	2.98
Neptune	450	165	2.99
Pluto*	590	248	2.99

The law of areas can be understood as a consequence of conservation of angular momentum whch is valid for any central force . A central force is such that the force on the planet is along the vector joining the Sun and the planet. Let the Sun be at the origin and let the position and momentum of the planet be denoted by **r** and **p** respectively. Then the area swept out by the planet of mass m in time interval Δt is (Fig. 8.2) $\Delta\mathbf{A}$ given by

$$\Delta\mathbf{A} = \tfrac{1}{2}\ (\mathbf{r} \times \mathbf{v}\Delta t) \qquad (8.1)$$

Hence

$$\Delta\mathbf{A}/\Delta t = \tfrac{1}{2}\,(\mathbf{r} \times \mathbf{p})/m, \text{ (since } \mathbf{v} = \mathbf{p}/m)$$
$$= \mathbf{L} / (2\ m) \qquad (8.2)$$

where **v** is the velocity, **L** is the angular momentum equal to (**r** × **p**). For a central force, which is directed along **r**, **L** is a constant as the planet goes around. Hence, $\Delta\mathbf{A}/\Delta t$ is a constant according to the last equation. This is the law of areas. Gravitation is a central force and hence the law of areas follows.

Johannes Kepler (1571–1630) was a scientist of German origin. He formulated the three laws of planetary motion based on the painstaking observations of Tycho Brahe and coworkers. Kepler himself was an assistant to Brahe and it took him sixteen long years to arrive at the three planetary laws. He is also known as the founder of geometrical optics, being the first to describe what happens to light after it enters a telescope.

* *Refer to information given in the Box on Page 182*

Example 8.1 Let the speed of the planet at the perihelion *P* in Fig. 8.1(a) be v_P and the Sun-planet distance SP be r_P. Relate $\{r_P, v_P\}$ to the corresponding quantities at the aphelion $\{r_A, v_A\}$. Will the planet take equal times to traverse *BAC* and *CPB* ?

Answer The magnitude of the angular momentum at *P* is $L_p = m_p r_p v_p$, since inspection tells us that $\mathbf{r}_p$ and $\mathbf{v}_p$ are mutually perpendicular. Similarly, $L_A = m_p r_A v_A$. From angular momentum conservation

$$m_p r_p v_p = m_p r_A v_A$$

or $$\frac{v_p}{v_A} = \frac{r_A}{r_p}$$

Since $r_A > r_p$, $v_p > v_A$.

The area *SBAC* bounded by the ellipse and the radius vectors *SB* and *SC* is larger than SBPC in Fig. 8.1. From Kepler's second law, equal areas are swept in equal times. Hence the planet will take a longer time to traverse *BAC* than *CPB*.

8.3 UNIVERSAL LAW OF GRAVITATION

Legend has it that observing an apple falling from a tree, Newton was inspired to arrive at an universal law of gravitation that led to an explanation of terrestrial gravitation as well as of Kepler's laws. Newton's reasoning was that the moon revolving in an orbit of radius R_m was subject to a centripetal acceleration due to earth's gravity of magnitude

$$a_m = \frac{V^2}{R_m} = \frac{4\pi^2 R_m}{T^2} \qquad (8.3)$$

where *V* is the speed of the moon related to the time period *T* by the relation $V = 2\pi R_m / T$. The time period *T* is about 27.3 days and R_m was already known then to be about 3.84×10^8m. If we substitute these numbers in Eq. (8.3), we get a value of a_m much smaller than the value of acceleration due to gravity g on the surface of the earth, arising also due to earth's gravitational attraction.

Central Forces

We know the time rate of change of the angular momentum of a single particle about the origin is

$$\frac{d\mathbf{l}}{dt} = \mathbf{r} \times \mathbf{F}$$

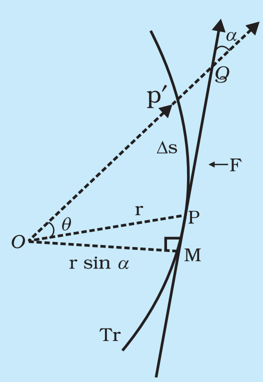
The angular momentum of the particle is conserved, if the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ due to the force $\mathbf{F}$ on it vanishes. This happens either when $\mathbf{F}$ is zero or when $\mathbf{F}$ is along $\mathbf{r}$. We are interested in forces which satisfy the latter condition. Central forces satisfy this condition. A 'central' force is always directed towards or away from a fixed point, i.e., along the position vector of the point of application of the force with respect to the fixed point. (See Figure below.) Further, the magnitude of a central force F depends on r, the distance of the point of application of the force from the fixed point; $F = F(r)$.

In the motion under a central force the angular momentum is always conserved. Two important results follow from this:

(1) The motion of a particle under the central force is always confined to a plane.
(2) The position vector of the particle with respect to the centre of the force (i.e. the fixed point) has a constant areal velocity. In other words the position vector sweeps out equal areas in equal times as the particle moves under the influence of the central force.

Try to prove both these results. You may need to know that the areal velocity is given by : $dA/dt = ½\, r\, v \sin\alpha$.

An immediate application of the above discussion can be made to the motion of a planet under the gravitational force of the sun. For convenience the sun may be taken to be so heavy that it is at rest. The gravitational force of the sun on the planet is directed towards the sun. This force also satisfies the requirement $F = F(r)$, since $F = G\, m_1 m_2/r^2$ where m_1 and m_2 are respectively the masses of the planet and the sun and G is the universal constant of gravitation. The two results (1) and (2) described above, therefore, apply to the motion of the planet. In fact, the result (2) is the well-known second law of Kepler.



Tr is the trejectory of the particle under the central force. At a position P, the force is directed along **OP**, *O is the centre of the force taken as the origin. In time Δt, the particle moves from P to P′, arc PP′ = Δs = v Δt. The tangent PQ at P to the trajectory gives the direction of the velocity at P. The area swept in Δt is the area of sector POP′ ≈ $(r \sin\alpha)$ PP′/2 = (r v sin α) Δt/2.)*

This clearly shows that the force due to earth's gravity decreases with distance. If one assumes that the gravitational force due to the earth decreases in proportion to the inverse square of the distance from the centre of the earth, we will have $a_m \alpha R_m^{-2}$; $g \alpha R_E^{-2}$ and we get

$$\frac{g}{a_m} = \frac{R_m^2}{R_E^2} \simeq 3600 \tag{8.4}$$

in agreement with a value of $g \simeq 9.8$ m s^{-2} and the value of a_m from Eq. (8.3). These observations led Newton to propose the following Universal Law of Gravitation :

Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The quotation is essentially from Newton's famous treatise called 'Mathematical Principles of Natural Philosophy' (Principia for short).

Stated Mathematically, Newton's gravitation law reads : The force **F** on a point mass m_2 due to another point mass m_1 has the magnitude

$$|\mathbf{F}| = G \frac{m_1 m_2}{r^2} \tag{8.5}$$

Equation (8.5) can be expressed in vector form as

$$\mathbf{F} = G \frac{m_1 m_2}{r^2}\left(-\hat{\mathbf{r}}\right) = -G \frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$$

$$= -G \frac{m_1 m_2}{|\mathbf{r}|^3}\hat{\mathbf{r}}$$

where G is the universal gravitational constant, $\hat{\mathbf{r}}$ is the unit vector from m_1 to m_2 and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ as shown in Fig. 8.3.

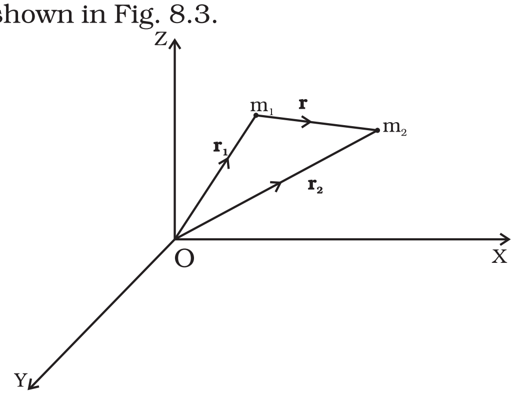


Fig. 8.3 *Gravitational force on m_1 due to m_2 is along* **r** *where the vector* **r** *is* $(\mathbf{r}_2 - \mathbf{r}_1)$.

The gravitational force is attractive, i.e., the force **F** is along – **r**. The force on point mass m_1 due to m_2 is of course – **F** by Newton's third law. Thus, the gravitational force $\mathbf{F}_{12}$ on the body 1 due to 2 and $\mathbf{F}_{21}$ on the body 2 due to 1 are related as $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Before we can apply Eq. (8.5) to objects under consideration, we have to be careful since the law refers to **point** masses whereas we deal with extended objects which have finite size. If we have a collection of point masses, the force on any one of them is the vector sum of the gravitational forces exerted by the other point masses as shown in Fig 8.4.

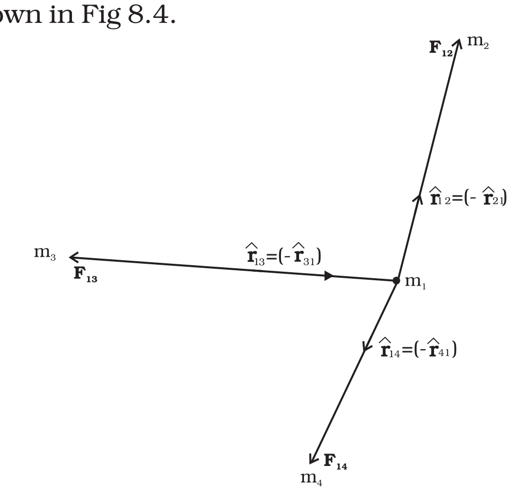


Fig. 8.4 *Gravitational force on point mass m_1 is the vector sum of the gravitational forces exerted by m_2, m_3 and m_4.*

The total force on m_1 is

$$\mathbf{F}_1 = \frac{Gm_2 m_1}{r_{21}^2}\hat{\mathbf{r}}_{21} + \frac{Gm_3 m_1}{r_{31}^2}\hat{\mathbf{r}}_{31} + \frac{Gm_4 m_1}{r_{41}^2}\hat{\mathbf{r}}_{41}$$

▶ ***Example 8.2*** Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.

(a) What is the force acting on a mass $2m$ placed at the centroid G of the triangle?

(b) What is the force if the mass at the vertex A is doubled ?

Take AG = BG = CG = 1 m (see Fig. 8.5)

Answer (a) The angle between GC and the positive x-axis is 30° and so is the angle between GB and the negative x-axis. The individual forces in vector notation are

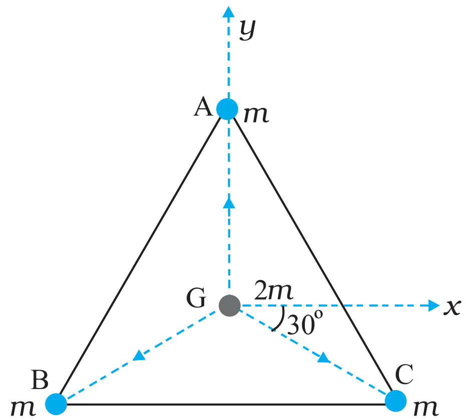


Fig. 8.5 *Three equal masses are placed at the three vertices of the Δ ABC. A mass 2m is placed at the centroid G.*

$$\mathbf{F}_{GA} = \frac{Gm(2m)}{1}\hat{\mathbf{j}}$$

$$\mathbf{F}_{GB} = \frac{Gm(2m)}{1}\left(-\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$

$$\mathbf{F}_{GC} = \frac{Gm(2m)}{1}\left(+\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$

From the principle of superposition and the law of vector addition, the resultant gravitational force $\mathbf{F}_R$ on $(2m)$ is

$$\mathbf{F}_R = \mathbf{F}_{GA} + \mathbf{F}_{GB} + \mathbf{F}_{GC}$$

$$\mathbf{F}_R = 2Gm^2\,\hat{\mathbf{j}} + 2Gm^2\left(-\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right)$$
$$+ 2Gm^2\left(\hat{\mathbf{i}}\cos 30^o - \hat{\mathbf{j}}\sin 30^o\right) = 0$$

Alternatively, one expects on the basis of symmetry that the resultant force ought to be zero.

(b) Now if the mass at vertex A is doubled then

$$F'_{GA} = \frac{G2m.2m}{1}\hat{j} = 4Gm^2\hat{j}$$

$$F'_{GB} = F_{GB} \textit{ and } F'_{GC} = F_{GC}$$

$$F'_R = F'_{GA} + F'_{GB} + F'_{GC}$$

$$F'_R = 2Gm^2\,\hat{j}$$

For the gravitational force between an extended object (like the earth) and a point mass, Eq. (8.5) is not directly applicable. Each point mass in the extended object will exert a force on the given point mass and these force will not all be in the same direction. We have to add up these forces vectorially for all the point masses in the extended object to get the total force. This is easily done using calculus. For two special cases, a simple law results when you do that :

(1) **The force of attraction between a hollow spherical shell of uniform density and a point mass situated outside is just as if the entire mass of the shell is concentrated at the centre of the shell.** Qualitatively this can be understood as follows: Gravitational forces caused by the various regions of the shell have components along the line joining the point mass to the centre as well as along a direction prependicular to this line. The components prependicular to this line cancel out when summing over all regions of the shell leaving only a resultant force along the line joining the point to the centre. The magnitude of this force works out to be as stated above.

Newton's Principia

Kepler had formulated his third law by 1619. The announcement of the underlying universal law of gravitation came about seventy years later with the publication in 1687 of Newton's masterpiece **Philosophiae Naturalis Principia Mathematica**, often simply called the **Principia**.

Around 1685, Edmund Halley (after whom the famous Halley's comet is named), came to visit Newton at Cambridge and asked him about the nature of the trajectory of a body moving under the influence of an inverse square law. Without hesitation Newton replied that it had to be an ellipse, and further that he had worked it out long ago around 1665 when he was forced to retire to his farm house from Cambridge on account of a plague outbreak. Unfortunately, Newton had lost his papers. Halley prevailed upon Newton to produce his work in book form and agreed to bear the cost of publication. Newton accomplished this feat in eighteen months of superhuman effort. The **Principia** is a singular scientific masterpiece and in the words of Lagrange it is "the greatest production of the human mind." The Indian born astrophysicist and Nobel laureate S. Chandrasekhar spent ten years writing a treatise on the **Principia**. His book, Newton's **Principia for the Common Reader** brings into sharp focus the beauty, clarity and breath taking economy of Newton's methods.