For example, the converse of the statement

p: If a number is divisible by 10*, it is divisible by* 5 *is*

q: If a number is divisible by 5*, then it is divisible by* 10.

Example 10 Write the converse of the following statements.

(i) If a number n is even, then n^2 is even.

(ii) If you do all the exercises in the book, you get an A grade in the class.

(iii) If two integers a and b are such that $a > b$, then $a - b$ is always a positive integer.

Solution The converse of these statements are

(i) If a number n^2 is even, then n is even.

(ii) If you get an A grade in the class, then you have done all the exercises of the book.

(iii) If two integers a and b are such that $a - b$ is always a positive integer, then $a > b$.

Let us consider some more examples.

Example 11 For each of the following compound statements, first identify the corresponding component statements. Then check whether the statements are true or not.

(i) If a triangle ABC is equilateral, then it is isosceles.

(ii) If a and b are integers, then ab is a rational number.

Solution (i) The component statements are given by

p : Triangle ABC is equilateral.

q : Triangle ABC is Isosceles.

Since an equilateral triangle is isosceles, we infer that the given compound statement is true.

(ii) The component statements are given by

p : a and b are integers.

q : ab is a rational number.

since the product of two integers is an integer and therefore a rational number, the compound statement is true.

'If and only if', represented by the symbol '$\Leftrightarrow$' means the following equivalent forms for the given statements p and q.

(i) p if and only if q

(ii) q if and only if p

(iii) p is necessary and sufficient condition for q and vice-versa

(iv) $p \Leftrightarrow q$

Consider an example.

Example 12 Given below are two pairs of statements. Combine these two statements using "if and only if ".

(i) *p:* If a rectangle is a square, then all its four sides are equal.

q: If all the four sides of a rectangle are equal, then the rectangle is a square.

(ii) *p:* If the sum of digits of a number is divisible by 3, then the number is divisible by 3.

q: If a number is divisible by 3, then the sum of its digits is divisible by 3.

Solution (i) A rectangle is a square if and only if all its four sides are equal.

(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

EXERCISE 14.4

1. Rewrite the following statement with "if-then" in five different ways conveying the same meaning.

If a natural number is odd, then its square is also odd.

2. Write the contrapositive and converse of the following statements.

(i) If x is a prime number, then x is odd.

(ii) If the two lines are parallel, then they do not intersect in the same plane.

(iii) Something is cold implies that it has low temperature.

(iv) You cannot comprehend geometry if you do not know how to reason deductively.

(v) x is an even number implies that x is divisible by 4.

3. Write each of the following statements in the form "if-then"

(i) You get a job implies that your credentials are good.

(ii) The Bannana trees will bloom if it stays warm for a month.

(iii) A quadrilateral is a parallelogram if its diagonals bisect each other.

(iv) To get an A^+ in the class, it is necessary that you do all the exercises of the book.

4. Given statements in (a) and (b). Identify the statements given below as contrapositive or converse of each other.

(a) If you live in Delhi, then you have winter clothes.

(i) If you do not have winter clothes, then you do not live in Delhi.

(ii) If you have winter clothes, then you live in Delhi.

(b) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(i) If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

(ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

14.6 Validating Statements

In this Section, we will discuss when a statement is true. To answer this question, one must answer all the following questions.

What does the statement mean? What would it mean to say that this statement is true and when this statement is not true?

The answer to these questions depend upon which of the special words and phrases “and”, “or”, and which of the implications “if and only”, “if-then”, and which of the quantifiers “for every”, “there exists”, appear in the given statement.

Here, we shall discuss some techniques to find when a statement is valid.

We shall list some general rules for checking whether a statement is true or not.

Rule 1 ***If p and q are mathematical statements, then in order to show that the statement “p and q” is true, the following steps are followed.***

Step-1 Show that the statement p is true.

Step-2 Show that the statement q is true.

Rule 2 ***Statements with “Or”***

If p and q are mathematical statements , then in order to show that the statement “p or q” is true, one must consider the following.

Case 1 By assuming that p is false, show that q must be true.

Case 2 By assuming that q is false, show that p must be true.

Rule 3 ***Statements with “If-then”***

In order to prove the statement "if p then q" we need to show that *any one* of the following case is true.

Case 1 By assuming that p is true, prove that q must be true.(Direct method)

Case 2 By assuming that q is false, prove that p must be false.(Contrapositive method)

Rule 4 ***Statements with "if and only if "***

In order to prove the statement "p if and only if q", we need to show.

(i) *If p is true, then q is true* and (ii) *If q is true, then p is true*

Now we consider some examples.

Example 13 Check whether the following statement is true or not.
If $x, y \in \mathbf{Z}$ are such that x and y are odd, then xy is odd.

Solution Let $p : x, y \in \mathbf{Z}$ such that x and y are odd

$$q : xy \text{ is odd}$$

To check the validity of the given statement, we apply Case 1 of Rule 3. That is assume that if p is true, then q is true.
p is true means that x and y are odd integers. Then

$x = 2m + 1$, for some integer m. $y = 2n + 1$, for some integer n. Thus

$$xy = (2m + 1)(2n + 1)$$
$$= 2(2mn + m + n) + 1$$

This shows that xy is odd. Therefore, the given statement is true.

Suppose we want to check this by using Case 2 of Rule 3, then we will proceed as follows.

We assume that q is not true. This implies that we need to consider the negation of the statement q. This gives the statement

$$\sim q : \text{Product } xy \text{ is even.}$$

This is possible only if either x or y is even. This shows that p is not true. Thus we have shown that

$$\sim q \Rightarrow \sim p$$

Note The above example illustrates that to prove $p \Rightarrow q$, it is enough to show $\sim q \Rightarrow \sim p$ which is the contrapositive of the statement $p \Rightarrow q$.

Example 14 Check whether the following statement is true or false by proving its contrapositive. If $x, y \in \mathbf{Z}$ such that xy is odd, then both x and y are odd.

Solution Let us name the statements as below *p : xy is odd.*

q : both x and y are odd.

We have to check whether the statement $p \Rightarrow q$ is true or not, that is, by checking its contrapositive statement i.e., $\sim q \Rightarrow \sim p$

Now $\sim q$: It is false that both x and y are odd. This implies that x (or y) is even.

Then $x = 2n$ for some integer n.

Therefore, $xy = 2ny$ for some integer n. This shows that xy is even. That is $\sim p$ is true. Thus, we have shown that $\sim q \Rightarrow \sim p$ and hence the given statement is true.

Now what happens when we combine an implication and its converse? Next, we shall discuss this.

Let us consider the following statements.

p : A tumbler is half empty.

q : A tumbler is half full.

We know that if the first statement happens, then the second happens and also if the second happens, then the first happens. We can express this fact as

If a tumbler is half empty, then it is half full.

If a tumbler is half full, then it is half empty.

We combine these two statements and get the following:

A tumbler is half empty if and only if it is half full.

Now, we discuss another method.

14.6.1 ***By Contradiction*** Here to check whether a statement p is true, we assume that p is not true i.e. $\sim p$ is true. Then, we arrive at some result which contradicts our assumption. Therefore, we conclude that p is true.

Example 15 Verify by the method of contradiction.

$$p: \sqrt{7} \text{ is irrational}$$

Solution In this method, we assume that the given statement is false. That is we assume that $\sqrt{7}$ is rational. This means that there exists positive integers a and b such that $\sqrt{7} = \frac{a}{b}$, where a and b have no common factors. Squaring the equation, we get $7 = \frac{a^2}{b^2} \Rightarrow a^2 = 7b^2 \Rightarrow 7$ divides a. Therefore, there exists an integer c such that $a = 7c$. Then $a^2 = 49c^2$ and $a^2 = 7b^2$

Hence, $7b^2 = 49c^2 \Rightarrow b^2 = 7c^2 \Rightarrow 7$ divides b. But we have already shown that 7 divides a. This implies that 7 is a common factor of both of a and b which contradicts our earlier assumption that a and b have no common factors. This shows that the assumption $\sqrt{7}$ is rational is wrong. Hence, the statement $\sqrt{7}$ is irrational is true.

Next, we shall discuss a method by which we may show that a statement is false. The method involves giving an ***example of a situation where the statement is not valid.*** Such an example is called a ***counter example***. The name itself suggests that this is an example to counter the given statement.

Example 16 By giving a counter example, show that the following statement is false. If n is an odd integer, then n is prime.

Solution The given statement is in the form "if p then q" we have to show that this is false. For this purpose we need to show that if p then $\sim q$. To show this we look for an odd integer n which is not a prime number. 9 is one such number. So $n = 9$ is a counter example. Thus, we conclude that the given statement is false.

In the above, we have discussed some techniques for checking whether a statement is true or not.

Note In mathematics, counter examples are used to disprove the statement. However, generating examples in favour of a statement do not provide validity of the statement.

EXERCISE 14.5

1. Show that the statement
p: "If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by
(i) direct method, (ii) method of contradiction, (iii) method of contrapositive

2. Show that the statement "For any real numbers a and b, $a^2 = b^2$ implies that $a = b$" is not true by giving a counter-example.

3. Show that the following statement is true by the method of contrapositive.
p: If x is an integer and x^2 is even, then x is also even.

4. By giving a counter example, show that the following statements are not true.
(i) p: If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle.
(ii) q: The equation $x^2 - 1 = 0$ does not have a root lying between 0 and 2.