14.4.3 *Quantifiers* Quantifiers are phrases like, "There exists" and "For all". Another phrase which appears in mathematical statements is "there exists". For example, consider the statement. *p*: *There exists a rectangle whose all sides are equal.* This means that there is atleast one rectangle whose all sides are equal.

A word closely connected with "there exists" is "for every" (or for all). Consider a statement.

p: For every prime number p, $\sqrt{p}$ *is an irrational number.*

This means that if S denotes the set of all prime numbers, then for all the members *p* of the set S, $\sqrt{p}$ is an irrational number.

In general, a mathematical statement that says "for every" can be interpreted as saying that all the members of the given set S where the property applies must satisfy that property.

We should also observe that it is important to know precisely where in the sentence a given connecting word is introduced. For example, compare the following two sentences:

1. For every positive number x there exists a positive number y such that $y < x$.
2. There exists a positive number y such that for every positive number x, we have $y < x$.

Although these statements may look similar, they do not say the same thing. As a matter of fact, (1) is true and (2) is false. Thus, in order for a piece of mathematical writing to make sense, all of the symbols must be carefully introduced and each symbol must be introduced precisely at the right place – not too early and not too late.

The words "And" and "Or" are called *connectives* and "There exists" and "For all" are called *quantifiers.*

Thus, we have seen that many mathematical statements contain some special words and it is important to know the meaning attached to them, especially when we have to check the validity of different statements.

EXERCISE 14.3

1. For each of the following compound statements first identify the connecting words and then break it into component statements.

(i) All rational numbers are real and all real numbers are not complex.

(ii) Square of an integer is positive or negative.

(iii) The sand heats up quickly in the Sun and does not cool down fast at night.

(iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

2. Identify the quantifier in the following statements and write the negation of the statements.

(i) There exists a number which is equal to its square.

(ii) For every real number x, x is less than $x + 1$.

(iii) There exists a capital for every state in India.

3. Check whether the following pair of statements are negation of each other. Give reasons for your answer.

(i) $x + y = y + x$ is true for every real numbers x and y.

(ii) There exists real numbers x and y for which $x + y = y + x$.

4. State whether the "Or" used in the following statements is "exclusive "or" inclusive. Give reasons for your answer.

(i) Sun rises or Moon sets.

(ii) To apply for a driving licence, you should have a ration card or a passport.

(iii) All integers are positive or negative.

14.5 Implications

In this Section, we shall discuss the implications of "if-then", "only if" and "if and only if".

The statements with "if-then" are very common in mathematics. For example, consider the statement.

r: If you are born in some country, then you are a citizen of that country.

When we look at this statement, we observe that it corresponds to two statements p and q given by

p : you are born in some country.

q : you are citizen of that country.

Then the sentence "if p then q" says that in the event if p is true, then q must be true.

One of the most important facts about the sentence "if p then q" is that it does not say any thing (or places no demand) on q when p is false. For example, if you are not born in the country, then you cannot say anything about q. To put it in other words" not happening of p has no effect on happening of q.

Another point to be noted for the statement "if p then q" is that the statement does not imply that p happens.

There are several ways of understanding "if p then q" statements. We shall illustrate these ways in the context of the following statement.

r: If a number is a multiple of 9, then it is a multiple of 3.

Let p and q denote the statements

p : a number is a multiple of 9.

q: a number is a multiple of 3.

Then, if p then q is the same as the following:

1. p **implies** q is denoted by $p \Rightarrow q$. The symbol $\Rightarrow$ stands for implies.
   This says that a number is a multiple of 9 implies that it is a multiple of 3.
2. p is a sufficient condition for q.
   This says that knowing that a number as a multiple of 9 is sufficient to conclude that it is a multiple of 3.
3. p only if q.
   This says that a number is a multiple of 9 only if it is a multiple of 3.
4. q is a necessary condition for p.
   This says that when a number is a multiple of 9, it is necessarily a multiple of 3.
5. $\sim q$ implies $\sim p$.
   This says that if a number is not a multiple of 3, then it is not a multiple of 9.

14.5.1 *Contrapositive and converse* Contrapositive and converse are certain other statements which can be formed from a given statement with "if-then".

For example, let us consider the following "if-then" statement.

If the physical environment changes, then the biological environment changes.

Then the contrapositive of this statement is

If the biological environment does not change, then the physical environment does not change.

Note that both these statements convey the same meaning.

To understand this, let us consider more examples.

Example 9 Write the contrapositive of the following statement:

(i) If a number is divisible by 9, then it is divisible by 3.

(ii) If you are born in India, then you are a citizen of India.

(iii) If a triangle is equilateral, it is isosceles.

Solution The contrapositive of the these statements are

(i) If a number is not divisible by 3, it is not divisible by 9.

(ii) If you are not a citizen of India, then you were not born in India.

(iii) If a triangle is not isosceles, then it is not equilateral.

The above examples show the contrapositive of the statement if p, then q is "if $\sim q$, then $\sim p$".

Next, we shall consider another term called *converse*.

The converse of a given statement "if p, then q" is if q, then p.