Question

A particle of unit mass is moving along the x-axis under the influence of a force and its total energy is conserved. Four possible forms of the potential energy of the particle are given in List I (a and U_0 are constants). Match the potential energies in List I to the corresponding statement(s) in List II.

List 1

$$U_1(x) = \frac{U_0}{2}\left[1 - \left(\frac{x}{a}\right)^2\right]^2$$

$$U_2(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2$$

$$U_3(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2 \exp\left[-\left(\frac{x}{a}\right)^2\right]$$

$$U_4(x) = \frac{U_0}{2}\left[\frac{x}{a} - \frac{1}{3}\left(\frac{x}{a}\right)^3\right]$$

List 2

The force acting on the particle is zero at $x = a$.

The force acting on the particle is zero at $x = 0$.

The force acting on the particle is zero at $x = -a$.

The particle experiences an attractive force towards $x = 0$ in the region $|x| < a$.

The particle with total energy $\frac{U_0}{4}$ can oscillate about the point $x = -a$.

Solution

For conservative force the force field is given as $F = -\frac{dU}{dx}$

A:

$$U_1(x) = \frac{U_0}{2}\left[1 - \left(\frac{x}{a}\right)^2\right]^2$$

Differentiating U w.r.t. x:

$$\Rightarrow F = -\frac{dU_1(x)}{dx} = -\frac{2U_0}{2}\left[1 - \left(\frac{x}{a}\right)^2\right]\left(0 - 2\left(\frac{x}{a^2}\right)\right)$$

$F = -\frac{2U_0}{a^4}x(x+a)(x-a)$Force is zero at x=+ a, x=-a and x=0, I, II and III applies to A.

$\Rightarrow$ For attractive force F α $-x$ For x near origin F is proportional to x. hence, (iv) does not apply

Calculating $\frac{d^2U}{dx^2}$ at x=-a

$\Rightarrow \frac{d^2U}{dx^2} = \frac{2U_0}{a^4}((x+a)(x-a) + x(x-a) + x(x+a))$ At x=-a the double derivative is positive which implies curve is having minima at this point and will oscillate about the point as it is minimum P.E. (v) and T.E. $\frac{U_o}{4}$ of particle is less than maximum P.E. $\frac{U_o}{2}$ applies Correct answers are I,II,III and V

B:

$$U_2(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2$$

Differentiating U w.r.t. x:$\Rightarrow F=-\frac{dU_2(x)}{dx} = -\frac{2U_0}{2}\left(\frac{x}{a^2}\right) = -\frac{U_0}{a^2}x$Force is zero at $x=0$, II applies to B

$\Rightarrow$For attractive force F α $-x$ For x near origin F α $-x$ (iv) applies

Calculating $\frac{d^2U}{dx^2}$ at x=-a

$\Rightarrow \frac{d^2U}{dx^2} = \frac{2U_0}{a^2}$ At x=-a the double derivative is positive and first derivative is non zero at this point and will not oscillate about the point as it is not a maximum or minimum P.E. (v) does not apply. Correct answers are II and IV

C:

$$U_3(x) = \frac{U_0}{2}\left(\frac{x}{a}\right)^2 \exp\left[-\left(\frac{x}{a}\right)\right]^2$$

$\Rightarrow F = -\frac{dU_3(x)}{dx} = \frac{U_0}{a^3}x(x-a)(x+a)\exp\left[-\left(\frac{x}{a}\right)\right]^2$Force is zero at $x=+a$, $x=-a$ and $x=0$, I, II and III applies to C

$\Rightarrow$ For attractive force F α $-x$ For near origin F is proportional to -x, hence, (iv) applies

Calculating $\frac{d^2U}{dx^2}$ at x=-a

$\Rightarrow \frac{d^2U}{dx^2} = -\frac{U_0}{a^3}[(x+a)(x-a)+x(x-a)-x(x+a)+x(x-a)(x+a)\frac{2x}{a}]\exp\left[-\left(\frac{x}{a}\right)\right]^2$ At x=-a the double derivative is negative which implies curve is having maxima at this point and will not oscillate about the point as it is maximum P.E. (v) does not apply.Correct answers are I,II, III and IV

D:

$$U_4(x) = \frac{U_0}{2}\left[\frac{x}{a} - \frac{1}{3}\left(\frac{x}{a}\right)^3\right]$$

$$\Rightarrow F = -\frac{U_0}{2a^3}\left(a^2 - x^2\right)$$

$F = -\frac{dU_4(x)}{dx} = \frac{U_0}{2a^3}(x-a)(x+a)$Force is zero at x=a and x=-a, I and III applies to D

$\Rightarrow$ For attractive force F α $-$xFor x<< a near origin F α const. (iv) does not apply

Calculating $\frac{d^2U}{dx^2}$ at x=-a$\Rightarrow$ $\frac{d^2U}{dx^2} = -\frac{U_0}{2a^3}[(x+a)+(x-a)]$ At x=-a the double derivative is positive which implies curve is having minima at this point and will oscillate about the point as it is minimum P.E. and T.E. $\frac{U_\circ}{4}$ of particle is less than maximum P.E. $\frac{U_\circ}{3}$(v) applies Correct answers are I, III and V