Q. The locus of the centre of the circle that externally touches the circles $x^2 + y^2 = 9$ and $(x-3)^2 + (y-4)^2 = 16$

[A] $x^2 + y^2 = (4x + 3y - 12)^2$

[B] $x^2 + y^2 = (3x + 4y - 12)^2$

[C] $x^2 + y^2 = \left(\frac{4x+3y-12}{5}\right)^2$

[D] $x^2 + y^2 = 25\left(\frac{3x+4y+12}{5}\right)^2$

Answer: [A]

Solution:

let

C_1: centre (h,k) & radius r

C_2: centre $(0,0)$ & radius 3

C_3: centre $(3,4)$ & radius 4

C_1C_2 touch externally implies

$$C_1C_2 = r_1 + r_2$$

$$\sqrt{h^2 + k^2} = r + 3 \quad \ldots [1]$$

C_1C_3 touch externally implies

$$C_1C_3 = r_1 + r_3$$

$$\sqrt{(h-3)^2 + (k-4)^2} = r + 4 \quad ...[2]$$

Solving equations [1] and [2]

$$\sqrt{(h-3)^2 + (k-4)^2} - \sqrt{h^2 + k^2} = 1$$

{ Note: difference of distances from 2 points is constant, ie the locus would be an half ellipse with its foci at (3,4) and (1,1) }

$$(h-3)^2 + (k-4)^2 = 1 + h^2 + k^2 + 2\sqrt{h^2 + k^2}$$

$$\sqrt{h^2 + k^2} = 12 - 4h - 3k$$

On squaring again and replacing h,k by x,y, we get

$$x^2 + y^2 = (4x + 3y - 12)^2$$