17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

9.5.3 *Linear differential equations*

A differential equation of the from

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$$

where, P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q} \qquad \text{... (1)}$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x)\frac{dy}{dx} + \mathrm{P}.(g(x))\, y = \mathrm{Q}\,.\, g(x) \qquad \text{... (2)}$$

Choose $g(x)$ in such a way that R.H.S. becomes a derivative of $y \cdot g(x)$.

i.e. $$g(x)\frac{dy}{dx} + \text{P}.\, g(x)\, y = \frac{d}{dx}[y \cdot g(x)]$$

or $$g(x)\frac{dy}{dx} + \text{P}.\, g(x)\, y = g(x)\frac{dy}{dx} + y\, g'(x)$$

$\Rightarrow$ $$\text{P}.\, g(x) = g'(x)$$

or $$\text{P} = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to x, we get

$$\int \text{P}dx = \int \frac{g'(x)}{g(x)}dx$$

or $$\int \text{P}\cdot dx = \log(g(x))$$

or $$\text{g}(x) = e^{\int \text{P}\,dx}$$

On multiplying the equation (1) by $g(x) = e^{\int \text{P}\,dx}$, the L.H.S. becomes the derivative of some function of x and y. This function $g(x) = e^{\int \text{P}\,dx}$ is called *Integrating Factor* (I.F.) of the given differential equation.

Substituting the value of $g(x)$ in equation (2), we get

$$e^{\int \text{P}\,dx}\frac{dy}{dx} + \text{P}e^{\int \text{P}\,dx}y = \text{Q}\cdot e^{\int \text{P}\,dx}$$

or $$\frac{d}{dx}\left(ye^{\int \text{P}dx}\right) = \text{Q}e^{\int \text{P}dx}$$

Integrating both sides with respect to x, we get

$$y\cdot e^{\int \text{P}dx} = \int\left(\text{Q}.e^{\int \text{P}\,dx}\right)dx$$

or $$y = e^{-\int \text{P}\,dx}.\int\left(\text{Q}.e^{\int \text{P}\,dx}\right)dx + \text{C}$$

which is the general solution of the differential equation.

Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor (I.F) = $e^{\int \mathrm{P}\,dx}$.

(iii) Write the solution of the given differential equation as

$$y\ (\text{I.F}) = \int (\mathrm{Q} \times \text{I.F})\, dx + \mathrm{C}$$

In case, the first order linear differential equation is in the form $\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$, where, P_1 and Q_1 are constants or functions of y only. Then I.F = $e^{\mathrm{P}_1 dy}$ and the solution of the differential equation is given by

$$x\ .\ (\text{I.F}) = \int (\mathrm{Q}_1 \times \text{I.F})\, dy + \mathrm{C}$$

Example 19 Find the general solution of the differential equation $\frac{dy}{dx} - y = \cos x$.

Solution Given differential equation is of the form

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}, \text{ where } \mathrm{P} = -1 \text{ and } \mathrm{Q} = \cos x$$

Therefore $\qquad \text{I.F} = e^{\int -1\, dx} = e^{-x}$

Multiplying both sides of equation by I.F, we get

$$e^{-x}\frac{dy}{dx} - e^{-x} y = e^{-x}\cos x$$

or

$$\frac{dy}{dx}\left(y e^{-x}\right) = e^{-x}\cos x$$

On integrating both sides with respect to x, we get

$$y e^{-x} = \int e^{-x}\cos x\, dx + \mathrm{C} \qquad \ldots (1)$$

Let $\qquad \mathrm{I} = \int e^{-x}\cos x\, dx$

$$= \cos x\left(\frac{e^{-x}}{-1}\right) - \int (-\sin x)\,(-e^{-x})\, dx$$

$$= -\cos x\, e^{-x} - \int \sin x\, e^{-x}\, dx$$

$$= -\cos x\, e^{-x} - \left[\sin x(-e^{-x}) - \int \cos x\,(-e^{-x})\, dx\right]$$

$$= -\cos x\, e^{-x} + \sin x\, e^{-x} - \int \cos x\, e^{-x}\, dx$$

or $$\text{I} = -e^{-x}\cos x + \sin x\, e^{-x} - \text{I}$$

or $$2\text{I} = (\sin x - \cos x)\, e^{-x}$$

or $$\text{I} = \frac{(\sin x - \cos x)e^{-x}}{2}$$

Substituting the value of I in equation (1), we get

$$ye^{-x} = \left(\frac{\sin x - \cos x}{2}\right)e^{-x} + \text{C}$$

or $$y = \left(\frac{\sin x - \cos x}{2}\right) + \text{C}e^{x}$$

which is the general solution of the given differential equation.

Example 20 Find the general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2\ (x \neq 0)$.

Solution The given differential equation is

$$x\frac{dy}{dx} + 2y = x^2 \qquad \text{... (1)}$$

Dividing both sides of equation (1) by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is a linear differential equation of the type $\frac{dy}{dx} + \text{P}y = \text{Q}$, where $\text{P} = \frac{2}{x}$ and $\text{Q} = x$.

So $$\text{I.F} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2\ [as\ e^{\log f(x)} = f(x)]$$

Therefore, solution of the given equation is given by

$$y \cdot x^2 = \int (x)(x^2)\, dx + \text{C} = \int x^3 dx + \text{C}$$

or $$y = \frac{x^2}{4} + \text{C}x^{-2}$$

which is the general solution of the given differential equation.

Example 21 Find the general solution of the differential equation $y\,dx - (x + 2y^2)\,dy = 0$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} - \frac{x}{y} = 2y$$

This is a linear differential equation of the type $\frac{dx}{dy} + P_1 x = Q_1$, where $P_1 = -\frac{1}{y}$ and $Q_1 = 2y$. Therefore $\text{I.F} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log(y)^{-1}} = \frac{1}{y}$

Hence, the solution of the given differential equation is

$$x\frac{1}{y} = \int (2y)\left(\frac{1}{y}\right) dy + C$$

or $$\frac{x}{y} = \int (2dy) + C$$

or $$\frac{x}{y} = 2y + C$$

or $$x = 2y^2 + Cy$$

which is a general solution of the given differential equation.

Example 22 Find the particular solution of the differential equation

$$\frac{dy}{dx} + y\cot x = 2x + x^2 \cot x \ (x \neq 0)$$

given that $y = 0$ when $x = \frac{\pi}{2}$.

Solution The given equation is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = \cot x$ and $Q = 2x + x^2 \cot x$. Therefore

$$\text{I.F} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Hence, the solution of the differential equation is given by

$$y \,.\, \sin x = \int (2x + x^2 \cot x) \sin x\, dx + C$$

or $$y \sin x = \int 2x \sin x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = \sin x \left(\frac{2x^2}{2}\right) - \int \cos x \left(\frac{2x^2}{2}\right) dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x - \int x^2 \cos x \, dx + \int x^2 \cos x \, dx + C$$

or $$y \sin x = x^2 \sin x + C \quad \text{... (1)}$$

Substituting $y = 0$ and $x = \frac{\pi}{2}$ in equation (1), we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + C$$

or $$C = \frac{-\pi^2}{4}$$

Substituting the value of C in equation (1), we get

$$y \sin x = x^2 \sin x - \frac{\pi^2}{4}$$

or $$y = x^2 - \frac{\pi^2}{4 \sin x} \quad (\sin x \neq 0)$$

which is the particular solution of the given differential equation.

Example 23 Find the equation of a curve passing through the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x coordinate (abscissa) and the product of the x coordinate and y coordinate (ordinate) of that point.

Solution We know that the slope of the tangent to the curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = x + xy$$

or $$\frac{dy}{dx} - xy = x \quad \text{... (1)}$$

This is a linear differential equation of the type $\frac{dy}{dx} + Py = Q$, where $P = -x$ and $Q = x$.

Therefore, $$\text{I.F} = e^{\int -x \, dx} = e^{\frac{-x^2}{2}}$$

Hence, the solution of equation is given by

$$y \cdot e^{\frac{-x^2}{2}} = \int (x)\left(e^{\frac{-x^2}{2}}\right) dx + \mathrm{C} \qquad \text{... (2)}$$

Let $$\mathrm{I} = \int (x)\, e^{\frac{-x^2}{2}}\, dx$$

Let $\dfrac{-x^2}{2} = t$, then $-\ x\ dx = dt$ or $x\ dx = -\ dt$.

Therefore, $$\mathrm{I} = -\int e^t\, dt = -e^t = -\,e^{\frac{-x^2}{2}}$$

Substituting the value of I in equation (2), we get

$$y\, e^{\frac{-x^2}{2}} = -e^{\frac{-x^2}{2}} + \mathrm{C}$$

or $$y = -1 + \mathrm{C}\, e^{\frac{x^2}{2}} \qquad \text{... (3)}$$

Now (3) represents the equation of family of curves. But we are interested in finding a particular member of the family passing through (0, 1). Substituting $x = 0$ and $y = 1$ in equation (3) we get

$$1 = -\ 1 + \mathrm{C}\ .\ e^0 \quad \text{or} \quad \mathrm{C} = 2$$

Substituting the value of C in equation (3), we get

$$y = -1 + 2\, e^{\frac{x^2}{2}}$$

which is the equation of the required curve.

EXERCISE 9.6

For each of the differential equations given in Exercises 1 to 12, find the general solution:

1. $\dfrac{dy}{dx} + 2y = \sin x$ **2.** $\dfrac{dy}{dx} + 3y = e^{-2x}$ **3.** $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$

4. $\dfrac{dy}{dx} + (\sec x)\, y = \tan x \left(0 \le x < \dfrac{\pi}{2}\right)$ **5.** $\cos^2 x \dfrac{dy}{dx} + y = \tan x \left(0 \le x < \dfrac{\pi}{2}\right)$

6. $x\dfrac{dy}{dx} + 2y = x^2 \log x$ **7.** $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x} \log x$

8. $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx \ (x \ne 0)$

9. $x\frac{dy}{dx}+y-x+xy\cot x=0\ (x\neq 0)$ **10.** $(x+y)\frac{dy}{dx}=1$

11. $y\,dx + (x - y^2)\,dy = 0$ **12.** $(x+3y^2)\frac{dy}{dx}=y\ (y>0)$.

For each of the differential equations given in Exercises 13 to 15, find a particular solution satisfying the given condition:

13. $\frac{dy}{dx}+2y\tan x=\sin x;\ y=0$ when $x=\frac{\pi}{3}$

14. $(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{1+x^2};\ y=0$ when $x=1$

15. $\frac{dy}{dx}-3y\cot x=\sin 2x;\ y=2$ when $x=\frac{\pi}{2}$

16. Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

17. Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

18. The Integrating Factor of the differential equation $x\frac{dy}{dx}-y=2x^2$ is

(A) e^{-x} (B) e^{-y} (C) $\frac{1}{x}$ (D) x

19. The Integrating Factor of the differential equation

$(1-y^2)\frac{dx}{dy}+yx = ay\,(-1<y<1)$ is

(A) $\frac{1}{y^2-1}$ (B) $\frac{1}{\sqrt{y^2-1}}$ (C) $\frac{1}{1-y^2}$ (D) $\frac{1}{\sqrt{1-y^2}}$

Miscellaneous Examples

Example 24 Verify that the function $y = c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$, where c_1, c_2 are arbitrary constants is a solution of the differential equation

$$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)y=0$$