20. In a bank, principal increases continuously at the rate of $r\%$ per year. Find the value of r if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

22. In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

23. The general solution of the differential equation $\dfrac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

9.5.2 *Homogeneous differential equations*

Consider the following functions in x and y

$F_1(x, y) = y^2 + 2xy,$ $F_2(x, y) = 2x - 3y,$

$F_3(x, y) = \cos\left(\dfrac{y}{x}\right),$ $F_4(x, y) = \sin x + \cos y$

If we replace x and y by λx and λy respectively in the above functions, for any nonzero constant λ, we get

$F_1(\lambda x, \lambda y) = \lambda^2 (y^2 + 2xy) = \lambda^2 F_1(x, y)$

$F_2(\lambda x, \lambda y) = \lambda (2x - 3y) = \lambda F_2(x, y)$

$F_3(\lambda x, \lambda y) = \cos\left(\dfrac{\lambda y}{\lambda x}\right) = \cos\left(\dfrac{y}{x}\right) = \lambda^0 F_3(x, y)$

$F_4(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n F_4(x, y)$, for any $n \in \mathbf{N}$

Here, we observe that the functions F_1, F_2, F_3 can be written in the form $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ but F_4 can not be written in this form. This leads to the following definition:

A function $F(x, y)$ is said to be *homogeneous function of degree n* if

$F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for any nonzero constant λ.

We note that in the above examples, F_1, F_2, F_3 are homogeneous functions of degree 2, 1, 0 respectively but F_4 is not a homogeneous function.

We also observe that

$$F_1(x, y) = x^2\left(\frac{y^2}{x^2}+\frac{2y}{x}\right) = x^2 h_1\left(\frac{y}{x}\right)$$

or

$$F_1(x, y) = y^2\left(1+\frac{2x}{y}\right) = y^2 h_2\left(\frac{x}{y}\right)$$

$$F_2(x, y) = x^1\left(2-\frac{3y}{x}\right) = x^1 h_3\left(\frac{y}{x}\right)$$

or

$$F_2(x, y) = y^1\left(2\frac{x}{y}-3\right) = y^1 h_4\left(\frac{x}{y}\right)$$

$$F_3(x, y) = x^0 \cos\left(\frac{y}{x}\right) = x^0\ h_5\left(\frac{y}{x}\right)$$

$$F_4(x, y) \neq x^n\ h_6\left(\frac{y}{x}\right), \text{ for any } n \in \mathbf{N}$$

or

$$F_4(x, y) \neq y^n\ h_7\left(\frac{x}{y}\right), \text{ for any } n \in \mathbf{N}$$

Therefore, a function F (x, y) is a homogeneous function of degree n if

$$F(x, y) = x^n g\left(\frac{y}{x}\right) \quad \text{or} \quad y^n h\left(\frac{x}{y}\right)$$

A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be *homogenous* if $F(x, y)$ is a homogenous function of degree zero.

To solve a homogeneous differential equation of the type

$$\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right) \qquad \text{... (1)}$$

We make the substitution $y = v\,.\,x$ ... (2)

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of $\frac{dy}{dx}$ from equation (3) in equation (1), we get

$$v + x\frac{dv}{dx} = g(v)$$

or $$x\frac{dv}{dx} = g(v) - v \quad \text{... (4)}$$

Separating the variables in equation (4), we get

$$\frac{dv}{g(v)-v} = \frac{dx}{x} \quad \text{... (5)}$$

Integrating both sides of equation (5), we get

$$\int \frac{dv}{g(v)-v} = \int \frac{1}{x}dx + C \quad \text{... (6)}$$

Equation (6) gives general solution (primitive) of the differential equation (1) when we replace v by $\frac{y}{x}$.

Note If the homogeneous differential equation is in the form $\frac{dx}{dy} = F(x, y)$ where, $F(x, y)$ is homogenous function of degree zero, then we make substitution $\frac{x}{y} = v$ i.e., $x = vy$ and we proceed further to find the general solution as discussed above by writing $\frac{dx}{dy} = F(x, y) = h\left(\frac{x}{y}\right)$.

Example 15 Show that the differential equation $(x - y)\frac{dy}{dx} = x + 2y$ is homogeneous and solve it.

Solution The given differential equation can be expressed as

$$\frac{dy}{dx} = \frac{x+2y}{x-y} \quad \text{... (1)}$$

Let $$F(x, y) = \frac{x+2y}{x-y}$$

Now $$F(\lambda x, \lambda y) = \frac{\lambda(x+2y)}{\lambda(x-y)} = \lambda^0 \cdot f(x, y)$$

Therefore, F(x, y) is a homogenous function of degree zero. So, the given differential equation is a homogenous differential equation.

Alternatively,

$$\frac{dy}{dx} = \left(\frac{1+\frac{2y}{x}}{1-\frac{y}{x}}\right) = g\left(\frac{y}{x}\right) \quad \text{... (2)}$$

R.H.S. of differential equation (2) is of the form $g\left(\frac{y}{x}\right)$ and so it is a homogeneous function of degree zero. Therefore, equation (1) is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \quad \text{... (3)}$$

Differentiating equation (3) with respect to, x we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{... (4)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1) we get

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

or
$$x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

or
$$x\frac{dv}{dx} = \frac{v^2+v+1}{1-v}$$

or
$$\frac{v-1}{v^2+v+1}dv = \frac{-dx}{x}$$

Integrating both sides of equation (5), we get

$$\int\frac{v-1}{v^2+v+1}dv = -\int\frac{dx}{x}$$

or
$$\frac{1}{2}\int\frac{2v+1-3}{v^2+v+1}dv = -\log|x| + C_1$$

or $$\frac{1}{2}\int\frac{2v+1}{v^2+v+1}dv-\frac{3}{2}\int\frac{1}{v^2+v+1}dv=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\int\frac{1}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dv=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|-\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)=-\log|x|+C_1$$

or $$\frac{1}{2}\log|v^2+v+1|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right)+C_1 \qquad \text{(Why?)}$$

Replacing v by $\frac{y}{x}$, we get

or $$\frac{1}{2}\log\left|\frac{y^2}{x^2}+\frac{y}{x}+1\right|+\frac{1}{2}\log x^2=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\frac{1}{2}\log\left|\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)x^2\right|=\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+C_1$$

or $$\log|(y^2+xy+x^2)|=2\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right)+2C_1$$

or $$\log|(x^2+xy+y^2)|=2\sqrt{3}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)+C$$

which is the general solution of the differential equation (1)

Example 16 Show that the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx}=y\cos\left(\frac{y}{x}\right)+x$ is homogeneous and solve it.

Solution The given differential equation can be written as

$$\frac{dy}{dx}=\frac{y\cos\left(\frac{y}{x}\right)+x}{x\cos\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

It is a differential equation of the form $\frac{dy}{dx} = F(x, y)$.

Here
$$F(x, y) = \frac{y\cos\left(\frac{y}{x}\right) + x}{x\cos\left(\frac{y}{x}\right)}$$

Replacing x by λx and y by λy, we get

$$F(\lambda x, \lambda y) = \frac{\lambda\left[y\cos\left(\frac{y}{x}\right) + x\right]}{\lambda\left(x\cos\frac{y}{x}\right)} = \lambda^0 [F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero.

Therefore, the given differential equation is a homogeneous differential equation. To solve it we make the substitution

$$y = vx \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{... (3)}$$

Substituting the value of y and $\frac{dy}{dx}$ in equation (1), we get

$$v + x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v}$$

or
$$x\frac{dv}{dx} = \frac{v\cos v + 1}{\cos v} - v$$

or
$$x\frac{dv}{dx} = \frac{1}{\cos v}$$

or
$$\cos v \; dv = \frac{dx}{x}$$

Therefore
$$\int \cos v \; dv = \int \frac{1}{x}\, dx$$

or $\sin v = \log |x| + \log |C|$

or $\sin v = \log |Cx|$

Replacing v by $\frac{y}{x}$, we get

$$\sin\left(\frac{y}{x}\right) = \log |Cx|$$

which is the general solution of the differential equation (1).

Example 17 Show that the differential equation $2y\,e^{\frac{x}{y}}dx + \left(y - 2x\,e^{\frac{x}{y}}\right)dy = 0$ is homogeneous and find its particular solution, given that, $x = 0$ when $y = 1$.

Solution The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2x\,e^{\frac{x}{y}} - y}{2y\,e^{\frac{x}{y}}} \qquad \text{... (1)}$$

Let $$F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$$

Then $$F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0 [F(x, y)]$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \qquad \text{... (2)}$$

Differentiating equation (2) with respect to y, we get

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get

$$v + y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v}$$

or $$y\frac{dv}{dy} = \frac{2v\,e^v - 1}{2e^v} - v$$

or $$y\frac{dv}{dy} = -\frac{1}{2e^v}$$

or $$2e^v\,dv = \frac{-dy}{y}$$

or $$\int 2e^v \cdot dv = -\int \frac{dy}{y}$$

or $$2\,e^v = -\log|y| + C$$

and replacing v by $\frac{x}{y}$, we get

$$2\,e^{\frac{x}{y}} + \log|y| = C \qquad \text{... (3)}$$

Substituting $x = 0$ and $y = 1$ in equation (3), we get

$$2\,e^0 + \log|1| = C \Rightarrow C = 2$$

Substituting the value of C in equation (3), we get

$$2\,e^{\frac{x}{y}} + \log|y| = 2$$

which is the particular solution of the given differential equation.

Example 18 Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$.

Solution We know that the slope of the tangent at any point on a curve is $\frac{dy}{dx}$.

Therefore, $$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

or $$\frac{dy}{dx} = \frac{1+\frac{y^2}{x^2}}{\frac{2y}{x}} \qquad \dots (1)$$

Clearly, (1) is a homogenous differential equation. To solve it we make substitution

$$y = vx$$

Differentiating $y = vx$ with respect to x, we get

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

or $$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

or $$x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\frac{2v}{1-v^2}dv = \frac{dx}{x}$$

or $$\frac{2v}{v^2-1}dv = -\frac{dx}{x}$$

Therefore $$\int \frac{2v}{v^2-1}dv = -\int \frac{1}{x}dx$$

or $$\log |v^2 - 1| = -\log |x| + \log |C_1|$$

or $$\log |(v^2 - 1)(x)| = \log |C_1|$$

or $$(v^2 - 1)\,x = \pm C_1$$

Replacing v by $\frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2} - 1\right)x = \pm C_1$$

or $$(y^2 - x^2) = \pm C_1\, x \text{ or } x^2 - y^2 = Cx$$

EXERCISE 9.5

In each of the Exercises 1 to 10, show that the given differential equation is homogeneous and solve each of them.

1. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$

2. $y' = \dfrac{x+y}{x}$

3. $(x - y)\, dy - (x + y)\, dx = 0$

4. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

5. $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$

6. $x\, dy - y\, dx = \sqrt{x^2 + y^2}\, dx$

7. $\left\{x\cos\left(\dfrac{y}{x}\right) + y\sin\left(\dfrac{y}{x}\right)\right\} y\, dx = \left\{y\sin\left(\dfrac{y}{x}\right) - x\cos\left(\dfrac{y}{x}\right)\right\} x\, dy$

8. $x\dfrac{dy}{dx} - y + x\sin\left(\dfrac{y}{x}\right) = 0$

9. $y\, dx + x\log\left(\dfrac{y}{x}\right) dy - 2x\, dy = 0$

10. $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right) dy = 0$

For each of the differential equations in Exercises from 11 to 15, find the particular solution satisfying the given condition:

11. $(x + y)\, dy + (x - y)\, dx = 0;\ y = 1$ when $x = 1$

12. $x^2\, dy + (xy + y^2)\, dx = 0;\ y = 1$ when $x = 1$

13. $\left[x\sin^2\left(\dfrac{y}{x}\right) - y\right] dx + x\, dy = 0;\ y = \dfrac{\pi}{4}$ when $x = 1$

14. $\dfrac{dy}{dx} - \dfrac{y}{x} + \text{cosec}\left(\dfrac{y}{x}\right) = 0;\ y = 0$ when $x = 1$

15. $2xy + y^2 - 2x^2\dfrac{dy}{dx} = 0;\ y = 2$ when $x = 1$

16. A homogeneous differential equation of the from $\dfrac{dx}{dy} = h\left(\dfrac{x}{y}\right)$ can be solved by making the substitution.

(A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$

17. Which of the following is a homogeneous differential equation?

(A) $(4x + 6y + 5)\, dy - (3y + 2x + 4)\, dx = 0$

(B) $(xy)\, dx - (x^3 + y^3)\, dy = 0$

(C) $(x^3 + 2y^2)\, dx + 2xy\, dy = 0$

(D) $y^2\, dx + (x^2 - xy - y^2)\, dy = 0$

9.5.3 *Linear differential equations*

A differential equation of the from

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x$$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^x$$

$$\frac{dy}{dx} + \left(\frac{y}{x \log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + \mathrm{P}_1 x = \mathrm{Q}_1$$

where, P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q} \qquad \text{... (1)}$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x)\frac{dy}{dx} + \mathrm{P}.(g(x))\, y = \mathrm{Q}\, .\, g(x) \qquad \text{... (2)}$$