Find the general solution of the DE:
$(x^3-3xy^2)dx = (y^3-3x^2y)dy$.

Solution:

Given, $(x^3-3xy^2)dx = (y^3-3x^2y)dy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y}$$

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^3-3v^2x^3}{v^3x^3-3vx^3} = \frac{1-3v^2}{v^3-3v}$$

$$x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v$$

$$= \frac{1-3v^2-v^4+3v^2}{v^2-3v} = \frac{1-v^4}{v(v^2-3)}$$

$$\Rightarrow \frac{v(v^2-3)}{1-v^4}dv = \frac{dx}{x}$$

on integrating,

$$\int \frac{v(v^2-3)}{1-v^4}dv = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{v(v^2-3)}{(1+v^2)(1-v^2)}dv = \int \frac{dx}{x}$$

So, $$\frac{v(v^2-3)}{(1+v^2)(1-v^2)} = \frac{v^3-3v}{(1+v^2)(1-v)(1+v)}$$

$$= \frac{A}{1-v} + \frac{B}{1+v} + \frac{Cv+D}{1+v^2}$$

$$\therefore v^3-3v = A(1+v)(1+v^2) + B(1-v)(1+v^2) + (Cv+D)(1-v^2)$$

$$\Rightarrow A = -\frac{1}{2},\ B = \frac{1}{2},\ C = -2,\ D = 0$$

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$$\therefore \int \frac{v^3-3v}{(1-v^2)(1+v^2)}\,dv = \int \frac{-1/2}{1-v}\,dv + \int \frac{1/2}{1+v}\,dv + \int \frac{-2v}{1+v^2}\,dv$$

$$= \frac{1}{2}\log|1-v| + \frac{1}{2}\log|1+v| - \log|1+v^2|$$

$$= \frac{1}{2}\log|1-v^2| - \log|1+v^2|$$

$$\therefore \frac{1}{2}\log|1-v^2| - \log|1+v^2| = \log|x| + \log|c'|$$

$$\Rightarrow \log\left|\frac{\sqrt{1-v^2}}{1+v^2}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \log\left|\frac{\sqrt{1-y^2/x^2}}{1+y^2/x^2}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \log\left|\frac{x\sqrt{x^2-y^2}}{x^2+y^2}\right| = \log|x| + \log|c'|$$

$$\Rightarrow \frac{x\sqrt{x^2-y^2}}{x^2+y^2} = c'x$$

$$\Rightarrow x\sqrt{x^2-y^2} = c'x(x^2+y^2)$$

$$\boxed{\therefore \sqrt{x^2-y^2} = c'(x^2+y^2)}$$

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