6.2 NOTIONS OF WORK AND KINETIC ENERGY: THE WORK-ENERGY THEOREM

The following relation for rectilinear motion under constant acceleration a has been encountered in Chapter 3,

$$v^2 - u^2 = 2\ as \quad (6.2)$$

where u and v are the initial and final speeds and s the distance traversed. Multiplying both sides by $m/2$, we have

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mas = Fs \quad (6.2a)$$

where the last step follows from Newton's Second Law. We can generalise Eq. (6.2) to three dimensions by employing vectors

$$v^2 - u^2 = 2\ \mathbf{a.d}$$

Here **a** and **d** are acceleration and displacement vectors of the object respectively.
Once again multiplying both sides by $m/2$, we obtain

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = m\ \mathbf{a.d} = \mathbf{F.d} \quad (6.2b)$$

The above equation provides a motivation for the definitions of work and kinetic energy. The left side of the equation is the difference in the quantity 'half the mass times the square of the speed' from its initial value to its final value. We call each of these quantities the 'kinetic energy', denoted by K. The right side is a product of the displacement and the component of the force along the displacement. This quantity is called 'work' and is denoted by W. Eq. (6.2b) is then

$$K_f - K_i = W \quad (6.3)$$

where K_i and K_f are respectively the initial and final kinetic energies of the object. Work refers to the force and the displacement over which it acts. **Work is done by a force on the body over a certain displacement**.

Equation (6.2) is also a special case of the work-energy (WE) theorem : **The change in kinetic energy of a particle is equal to the work done on it by the net force**. We shall generalise the above derivation to a varying force in a later section.

Example 6.2 It is well known that a raindrop falls under the influence of the downward gravitational force and the opposing resistive force. The latter is known to be proportional to the speed of the drop but is otherwise undetermined. Consider a drop of mass 1.00 g falling from a height 1.00 km. It hits the ground with a speed of 50.0 m s^{-1}. (a) What is the work done by the gravitational force ? What is the work done by the unknown resistive force?

Answer (a) The change in kinetic energy of the drop is

$$\Delta K = \frac{1}{2}m\,v^2 - 0$$

$$= \frac{1}{2} \times 10^{-3} \times 50 \times 50$$

$$= 1.25\ \text{J}$$

where we have assumed that the drop is initially at rest.

Assuming that g is a constant with a value 10 m/s^2, the work done by the gravitational force is,

$$W_g = mgh$$
$$= 10^{-3} \times 10 \times 10^3$$
$$= 10.0\ \text{J}$$

(b) From the work-energy theorem

$$\Delta K = W_g + W_r$$

where W_r is the work done by the resistive force on the raindrop. Thus

$$W_r = \Delta K - W_g$$
$$= 1.25 - 10$$
$$= -8.75\ \text{J}$$

is negative. ◀

6.3 WORK

As seen earlier, work is related to force and the displacement over which it acts. Consider a constant force **F** acting on an object of mass m. The object undergoes a displacement **d** in the positive x-direction as shown in Fig. 6.2.

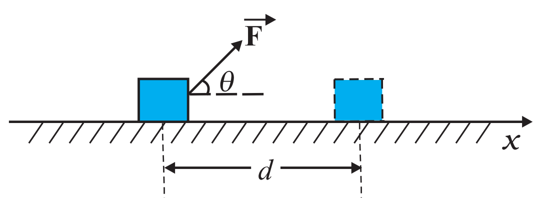


Fig. 6.2 *An object undergoes a displacement* **d** *under the influence of the force* **F**.

The work done by the force is defined to be the product of component of the force in the direction of the displacement and the magnitude of this displacement. Thus

$$W = (F\cos\theta)d = \mathbf{F.d} \qquad (6.4)$$

We see that if there is no displacement, there is no work done even if the force is large. Thus, when you push hard against a rigid brick wall, the force you exert on the wall does no work. Yet your muscles are alternatively contracting and relaxing and internal energy is being used up and you do get tired. Thus, the meaning of work in physics is different from its usage in everyday language.

No work is done if :

(i) the displacement is zero as seen in the example above. A weightlifter holding a 150 kg mass steadily on his shoulder for 30 s does no work on the load during this time.

(ii) the force is zero. A block moving on a smooth horizontal table is not acted upon by a horizontal force (since there is no friction), but may undergo a large displacement.

(iii) the force and displacement are mutually perpendicular. This is so since, for $\theta = \pi/2$ rad (= 90°), cos (π/2) = 0. For the block moving on a smooth horizontal table, the gravitational force *mg* does no work since it acts at right angles to the displacement. If we assume that the moon's orbits around the earth is perfectly circular then the earth's gravitational force does no work. The moon's instantaneous displacement is tangential while the earth's force is radially inwards and $\theta = \pi/2$.

Work can be both positive and negative. If θ is between 0° and 90°, cos θ in Eq. (6.4) is positive. If θ is between 90° and 180°, cos θ is negative. In many examples the frictional force opposes displacement and θ = 180°. Then the work done by friction is negative (cos 180° = –1).

From Eq. (6.4) it is clear that work and energy have the same dimensions, $[ML^2T^{-2}]$. The SI unit of these is joule (J), named after the famous British physicist James Prescott Joule (1811-1869). Since work and energy are so widely used as physical concepts, alternative units abound and some of these are listed in Table 6.1.

Table 6.1 Alternative Units of Work/Energy in J

erg	10^{-7} J
electron volt (eV)	1.6×10^{-19} J
calorie (cal)	4.186 J
kilowatt hour (kWh)	3.6×10^{6} J

▶ ***Example 6.3*** A cyclist comes to a skidding stop in 10 m. During this process, the force on the cycle due to the road is 200 N and is directly opposed to the motion. (a) How much work does the road do on the cycle ? (b) How much work does the cycle do on the road ?

Answer Work done on the cycle by the road is the work done by the stopping (frictional) force on the cycle due to the road.

(a) The stopping force and the displacement make an angle of 180° (π rad) with each other. Thus, work done by the road,

$$W_r = Fd\cos\theta$$
$$= 200 \times 10 \times \cos\pi$$
$$= -2000 \text{ J}$$

It is this negative work that brings the cycle to a halt in accordance with WE theorem.

(b) From Newton's Third Law an equal and opposite force acts on the road due to the cycle. Its magnitude is 200 N. However, the road undergoes no displacement. Thus, work done by cycle on the road is zero. ◀

The lesson of Example 6.3 is that though the force on a body A exerted by the body B is always equal and opposite to that on B by A (Newton's Third Law); the work done on A by B is not necessarily equal and opposite to the work done on B by A.

6.4 KINETIC ENERGY

As noted earlier, if an object of mass *m* has velocity **v**, its kinetic energy *K* is

$$K = \frac{1}{2}m\,\mathbf{v.v} = \frac{1}{2}mv^2 \qquad (6.5)$$

Kinetic energy is a scalar quantity. The kinetic energy of an object is a measure of the work an

Table 6.2 Typical kinetic energies (*K*)

Object	Mass (kg)	Speed (m s^{-1})	*K* (J)
Car	2000	25	6.3×10^{5}
Running athlete	70	10	3.5×10^{3}
Bullet	5×10^{-2}	200	10^{3}
Stone dropped from 10 m	1	14	10^{2}
Rain drop at terminal speed	3.5×10^{-5}	9	1.4×10^{-3}
Air molecule	$\simeq10^{-26}$	500	$\simeq10^{-21}$

object can do by the virtue of its motion. This notion has been intuitively known for a long time. The kinetic energy of a fast flowing stream has been used to grind corn. Sailing ships employ the kinetic energy of the wind. Table 6.2 lists the kinetic energies for various objects.

▶ ***Example 6.4*** In a ballistics demonstration a police officer fires a bullet of mass 50.0 g with speed 200 m s^{-1} (see Table 6.2) on soft plywood of thickness 2.00 cm. The bullet emerges with only 10% of its initial kinetic energy. What is the emergent speed of the bullet ?

Answer The initial kinetic energy of the bullet is $mv^2/2$ = 1000 J. It has a final kinetic energy of 0.1×1000 = 100 J. If v_f is the emergent speed of the bullet,

$$\frac{1}{2}mv_f^2 = 100 \text{ J}$$

$$v_f = \sqrt{\frac{2\times100 \text{ J}}{0.05 \text{ kg}}}$$

$$= 63.2 \text{ m s}^{-1}$$

The speed is reduced by approximately 68% (not 90%). ◀

6.5 WORK DONE BY A VARIABLE FORCE

A constant force is rare. It is the variable force, which is more commonly encountered. Fig. 6.3 is a plot of a varying force in one dimension.

If the displacement Δx is small, we can take the force $F(x)$ as approximately constant and the work done is then

$$\Delta W = F(x)\,\Delta x$$

This is illustrated in Fig. 6.3(a). Adding successive rectangular areas in Fig. 6.3(a) we get the total work done as

$$W \cong \sum_{x_i}^{x_f} F(x)\Delta x \qquad (6.6)$$

where the summation is from the initial position x_i to the final position x_f.

If the displacements are allowed to approach zero, then the number of terms in the sum increases without limit, but the sum approaches a definite value equal to the area under the curve in Fig. 6.3(b). Then the work done is

$$W = \lim_{\Delta x \to 0} \sum_{x_i}^{x_f} F(x)\Delta x$$

$$= \int_{x_i}^{x_f} F(x)\,\mathrm{d}x \qquad (6.7)$$

where '*lim*' stands for the limit of the sum when Δx tends to zero. Thus, for a varying force the work done can be expressed as a definite integral of force over displacement (see also Appendix 3.1).

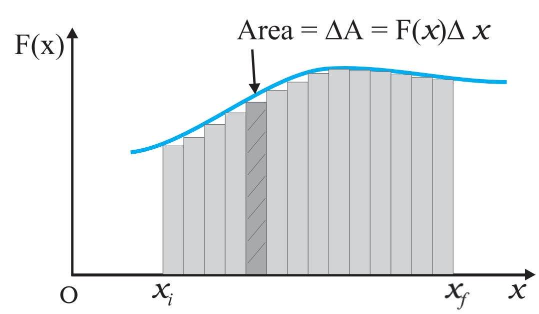
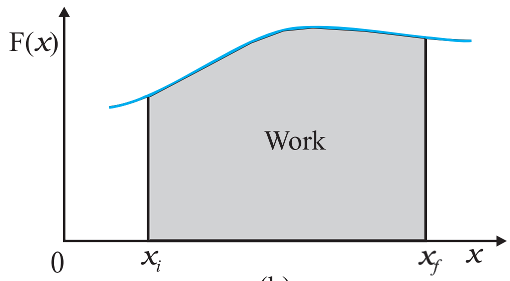


Fig. 6.3(a)



(b)

Fig. 6.3 *(a) The shaded rectangle represents the work done by the varying force F(x), over the small displacement Δx, ΔW = F(x) Δx. (b) adding the areas of all the rectangles we find that for Δx → 0, the area under the curve is exactly equal to the work done by F(x).*

▶ ***Example 6.5*** A woman pushes a trunk on a railway platform which has a rough surface. She applies a force of 100 N over a distance of 10 m. Thereafter, she gets progressively tired and her applied force reduces linearly with distance to 50 N. The total distance through which the trunk has been moved is 20 m. Plot the force applied by the woman and the frictional force, which is 50 N versus displacement. Calculate the work done by the two forces over 20 m.

Answer

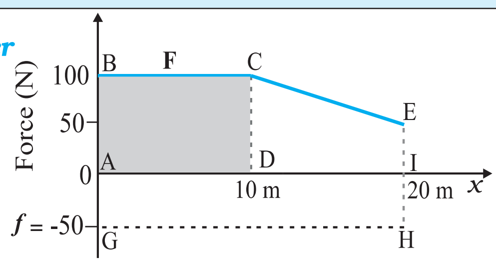


Fig. 6.4 *Plot of the force F applied by the woman and the opposing frictional force f versus displacement.*

The plot of the applied force is shown in Fig. 6.4. At x = 20 m, F = 50 N (≠ 0). We are given that the frictional force f is $|\mathbf{f}|$ = 50 N. It opposes motion and acts in a direction opposite to **F**. It is therefore, shown on the negative side of the force axis.

The work done by the woman is

$W_F \rightarrow$ area of the rectangle ABCD + area of the trapezium CEID

$$W_F = 100 \times 10 + \frac{1}{2}(100 + 50) \times 10$$

$$= 1000 + 750$$

$$= 1750 \text{ J}$$

The work done by the frictional force is

$W_f \rightarrow$ area of the rectangle AGHI

$$W_f = (-50) \times 20$$

$$= -1000 \text{ J}$$

The area on the negative side of the force axis has a negative sign. ◀

6.6 THE WORK-ENERGY THEOREM FOR A VARIABLE FORCE

We are now familiar with the concepts of work and kinetic energy to prove the work-energy theorem for a variable force. We confine ourselves to one dimension. The time rate of change of kinetic energy is

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2} m\, v^2\right)$$

$$= m \frac{dv}{dt} v$$

$$= F\, v \text{ (from Newton's Second Law)}$$

$$= F \frac{dx}{dt}$$

Thus

$$dK = F dx$$

Integrating from the initial position (x_i) to final position (x_f), we have

$$\int_{K_i}^{K_f} dK = \int_{x_i}^{x_f} F dx$$

where, K_i and K_f are the initial and final kinetic energies corresponding to x_i and x_f.

$$\text{or} \quad K_f - K_i = \int_{x_i}^{x_f} F dx \qquad (6.8a)$$

From Eq. (6.7), it follows that

$$K_f - K_i = W \qquad (6.8b)$$

Thus, the WE theorem is proved for a variable force.

While the WE theorem is useful in a variety of problems, it does not, in general, incorporate the complete dynamical information of Newton's second law. It is an integral form of Newton's second law. Newton's second law is a relation between acceleration and force at any instant of time. Work-energy theorem involves an integral over an interval of time. In this sense, the temporal (time) information contained in the statement of Newton's second law is 'integrated over' and is not available explicitly. Another observation is that Newton's second law for two or three dimensions is in vector form whereas the work-energy theorem is in scalar form. In the scalar form, information with respect to directions contained in Newton's second law is not present.

▶ ***Example 6.6*** A block of mass m = 1 kg, moving on a horizontal surface with speed v_i = 2 m s^{-1} enters a rough patch ranging from x = 0.10 m to x = 2.01 m. The retarding force F_r on the block in this range is inversely proportional to x over this range,

$$F_r = \frac{-k}{x} \text{ for } 0.1 < x < 2.01 \text{ m}$$

$= 0$ for $x < 0.1$m and $x > 2.01$ m

where $k = 0.5$ J. What is the final kinetic energy and speed v_f of the block as it crosses this patch ?

Answer From Eq. (6.8a)

$$K_f = K_i + \int_{0.1}^{2.01} \frac{(-k)}{x} \mathrm{d}x$$

$$= \frac{1}{2} m v_i^2 - k \ln(x)\Big|_{0.1}^{2.01}$$

$$= \frac{1}{2} m v_i^2 - k \ln(2.01/0.1)$$

$$= 2 - 0.5 \ln(20.1)$$

$$= 2 - 1.5 = 0.5 \text{ J}$$

$$v_f = \sqrt{2K_f/m} = 1 \text{ m s}^{-1}$$

Here, note that ln is a symbol for the natural logarithm to the base e and not the logarithm to the base 10 [$\ln X = \log_e X = 2.303 \log_{10} X$]. ◀

6.7 THE CONCEPT OF POTENTIAL ENERGY

The word potential suggests possibility or capacity for action. The term potential energy brings to one's mind 'stored' energy. A stretched bow-string possesses potential energy. When it is released, the arrow flies off at a great speed. The earth's crust is not uniform, but has discontinuities and dislocations that are called fault lines. These fault lines in the earth's crust are like 'compressed springs'. They possess a large amount of potential energy. An earthquake results when these fault lines readjust. Thus, potential energy is the 'stored energy' by virtue of the position or configuration of a body. The body left to itself releases this stored energy in the form of kinetic energy. Let us make our notion of potential energy more concrete.

The gravitational force on a ball of mass m is mg . g may be treated as a constant near the earth surface. By 'near' we imply that the height h of the ball above the earth's surface is very small compared to the earth's radius R_E ($h \ll R_E$) so that we can ignore the variation of g near the earth's surface*. In what follows we have taken the upward direction to be positive. Let us raise the ball up to a height h. The work done by the external agency against the gravitational force is mgh. This work gets stored as potential energy. Gravitational potential energy of an object, as a function of the height h, is denoted by $V(h)$ and it is the negative of work done by the gravitational force in raising the object to that height.

$$V(h) = mgh$$

If h is taken as a variable, it is easily seen that the gravitational force F equals the negative of the derivative of $V(h)$ with respect to h. Thus,

$$F = -\frac{\mathrm{d}}{\mathrm{d}h} V(h) = -m g$$

The negative sign indicates that the gravitational force is downward. When released, the ball comes down with an increasing speed. Just before it hits the ground, its speed is given by the kinematic relation,

$$v^2 = 2gh$$

This equation can be written as

$$\frac{1}{2} m v^2 = m g h$$

which shows that the gravitational potential energy of the object at height h, when the object is released, manifests itself as kinetic energy of the object on reaching the ground.

Physically, the notion of potential energy is applicable only to the class of forces where work done against the force gets 'stored up' as energy. When external constraints are removed, it manifests itself as kinetic energy. Mathematically, (for simplicity, in one dimension) the potential

* *The variation of g with height is discussed in Chapter 8 on Gravitation.*

energy $V(x)$ is defined if the force $F(x)$ can be written as

$$F(x) = -\frac{dV}{dx}$$

This implies that

$$\int_{x_i}^{x_f} F(x)dx = -\int_{V_i}^{V_f} dV = V_i - V_f$$

The work done by a conservative force such as gravity depends on the initial and final positions only. In the previous chapter we have worked on examples dealing with inclined planes. If an object of mass m is released from rest, from the top of a smooth (frictionless) inclined plane of height h, its speed at the bottom is $\sqrt{2gh}$ irrespective of the angle of inclination. Thus, at the bottom of the inclined plane it acquires a kinetic energy, mgh. If the work done or the kinetic energy did depend on other factors such as the velocity or the particular path taken by the object, the force would be called non-conservative.

The dimensions of potential energy are $[ML^2T^{-2}]$ and the unit is joule (J), the same as kinetic energy or work. To reiterate, the change in potential energy, for a conservative force, ΔV is equal to the negative of the work done by the force

$$\Delta V = -F(x)\,\Delta x \qquad (6.9)$$

In the example of the falling ball considered in this section we saw how potential energy was converted to kinetic energy. This hints at an important principle of conservation in mechanics, which we now proceed to examine.

6.8 THE CONSERVATION OF MECHANICAL ENERGY

For simplicity we demonstrate this important principle for one-dimensional motion. Suppose that a body undergoes displacement Δx under the action of a conservative force F. Then from the WE theorem we have,

$$\Delta K = F(x)\,\Delta x$$

If the force is conservative, the potential energy function $V(x)$ can be defined such that

$$-\Delta V = F(x)\,\Delta x$$

The above equations imply that

$$\Delta K + \Delta V = 0$$

$$\Delta(K + V) = 0 \qquad (6.10)$$

which means that $K + V$, the sum of the kinetic and potential energies of the body is a constant. Over the whole path, x_i to x_f, this means that

$$K_i + V(x_i) = K_f + V(x_f) \qquad (6.11)$$

The quantity $K + V(x)$, is called the total mechanical energy of the system. Individually the kinetic energy K and the potential energy $V(x)$ may vary from point to point, but the sum is a constant. The aptness of the term 'conservative force' is now clear.

Let us consider some of the definitions of a conservative force.

- A force $F(x)$ is conservative if it can be derived from a scalar quantity $V(x)$ by the relation given by Eq. (6.9). The three-dimensional generalisation requires the use of a vector derivative, which is outside the scope of this book.
- The work done by the conservative force depends only on the end points. This can be seen from the relation,
  $W = K_f - K_i = V(x_i) - V(x_f)$
  which depends on the end points.
- A third definition states that the work done by this force in a closed path is zero. This is once again apparent from Eq. (6.11) since $x_i = x_f$.

Thus, the principle of conservation of total mechanical energy can be stated as

The total mechanical energy of a system is conserved if the forces, doing work on it, are conservative.

The above discussion can be made more concrete by considering the example of the gravitational force once again and that of the spring force in the next section. Fig. 6.5 depicts a ball of mass m being dropped from a cliff of height H.

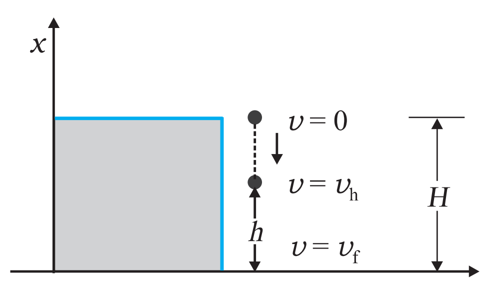


Fig. 6.5 *The conversion of potential energy to kinetic energy for a ball of mass m dropped from a height H.*

The total mechanical energies E_0, E_h, and E_H of the ball at the indicated heights zero (ground level), h and H, are

$$E_H = mgH \tag{6.11 a}$$

$$E_h = mgh + \frac{1}{2}mv_h^2 \tag{6.11 b}$$

$$E_0 = (1/2)\, mv_f^2 \tag{6.11 c}$$

The constant force is a special case of a spatially dependent force $F(x)$. Hence, the mechanical energy is conserved. Thus

$$E_H = E_0$$

or, $$mgH = \frac{1}{2}mv_f^2$$

$$v_f = \sqrt{2gH}$$

a result that was obtained in section 3.7 for a freely falling body.

Further,

$$E_H = E_h$$

which implies,

$$v_h^2 = 2g(H - h) \tag{6.11 d}$$

and is a familiar result from kinematics.

At the height H, the energy is purely potential. It is partially converted to kinetic at height h and is fully kinetic at ground level. This illustrates the conservation of mechanical energy.

▶ ***Example 6.7*** A bob of mass m is suspended by a light string of length L. It is imparted a horizontal velocity v_o at the lowest point A such that it completes a semi-circular trajectory in the vertical plane with the string becoming slack only on reaching the topmost point, C. This is shown in Fig. 6.6. Obtain an expression for (i) v_o; (ii) the speeds at points B and C; (iii) the ratio of the kinetic energies (K_B/K_C) at B and C. Comment on the nature of the trajectory of the bob after it reaches the point C.

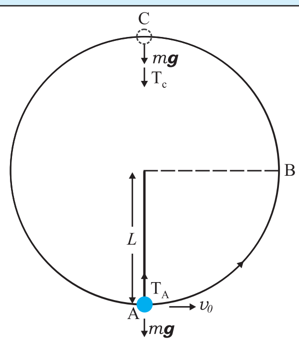


Fig. 6.6

Answer (i) There are two external forces on the bob : gravity and the tension (T) in the string. The latter does no work since the displacement of the bob is always normal to the string. The potential energy of the bob is thus associated with the gravitational force only. The total mechanical energy E of the system is conserved. We take the potential energy of the system to be zero at the lowest point A. Thus, at A :

$$E = \frac{1}{2}mv_0^2 \tag{6.12}$$

$$T_A - mg = \frac{mv_0^2}{L} \quad \text{[Newton's Second Law]}$$

where T_A is the tension in the string at A. At the highest point C, the string slackens, as the tension in the string (T_C) becomes zero.

Thus, at C

$$E = \frac{1}{2}mv_c^2 + 2mgL \tag{6.13}$$

$$mg = \frac{mv_c^2}{L} \quad \text{[Newton's Second Law]} \tag{6.14}$$

where v_C is the speed at C. From Eqs. (6.13) and (6.14)

$$E = \frac{5}{2}mgL$$

Equating this to the energy at A

$$\frac{5}{2}mgL = \frac{m}{2}v_0^2$$

or, $$v_0 = \sqrt{5gL}$$

(ii) It is clear from Eq. (6.14)

$$v_C = \sqrt{gL}$$

At B, the energy is

$$E = \frac{1}{2}mv_B^2 + mgL$$

Equating this to the energy at A and employing the result from (i), namely $v_0^2 = 5gL$,

$$\frac{1}{2}mv_B^2 + mgL = \frac{1}{2}mv_0^2$$

$$= \frac{5}{2}m\,g\,L$$

$$\therefore v_B = \sqrt{3gL}$$

(iii) The ratio of the kinetic energies at B and C is :

$$\frac{K_B}{K_C} = \frac{\frac{1}{2}mv_B^2}{\frac{1}{2}mv_C^2} = \frac{3}{1}$$

At point C, the string becomes slack and the velocity of the bob is horizontal and to the left. If the connecting string is cut at this instant, the bob will execute a projectile motion with horizontal projection akin to a rock kicked horizontally from the edge of a cliff. Otherwise the bob will continue on its circular path and complete the revolution. ◀

6.9 THE POTENTIAL ENERGY OF A SPRING

The spring force is an example of a variable force which is conservative. Fig. 6.7 shows a block attached to a spring and resting on a smooth horizontal surface. The other end of the spring is attached to a rigid wall. The spring is light and may be treated as massless. In an ideal spring, the spring force F_s is proportional to x where x is the displacement of the block from the equilibrium position. The displacement could be either positive [Fig. 6.7(b)] or negative [Fig. 6.7(c)]. This force law for the spring is called Hooke's law and is mathematically stated as

$$F_s = -kx$$

The constant k is called the spring constant. Its unit is N m^{-1}. The spring is said to be stiff if k is large and soft if k is small.

Suppose that we pull the block outwards as in Fig. 6.7(b). If the extension is x_m, the work done by the spring force is

$$W_s = \int_0^{x_m} F_s \, dx = -\int_0^{x_m} kx \, dx$$

$$= -\frac{k x_m^2}{2} \qquad (6.15)$$

This expression may also be obtained by considering the area of the triangle as in Fig. 6.7(d). Note that the work done by the external pulling force F is positive since it overcomes the spring force.

$$W = +\frac{k x_m^2}{2} \qquad (6.16)$$

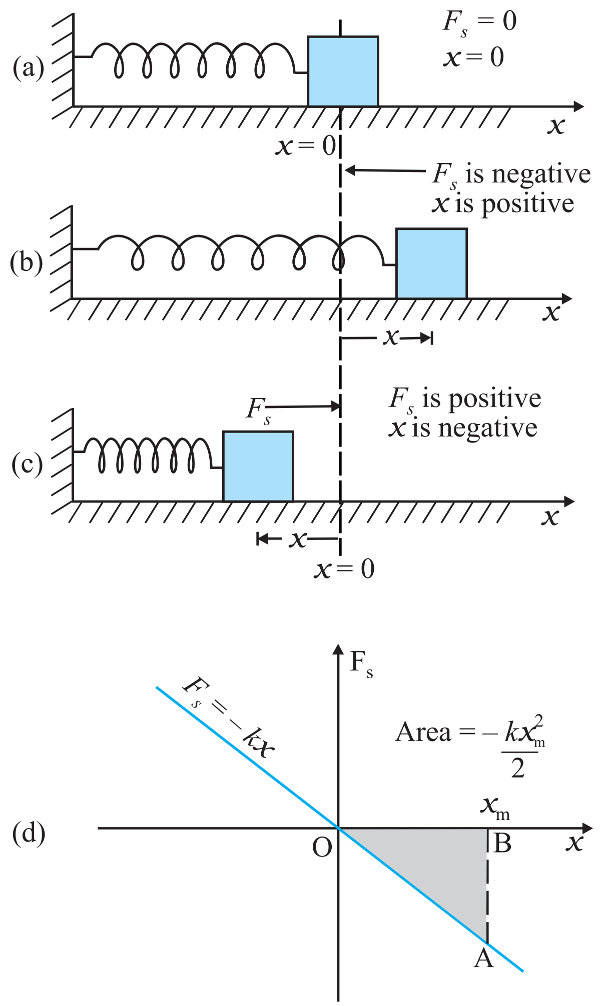


Fig. 6.7 *Illustration of the spring force with a block attached to the free end of the spring. (a) The spring force F_s is zero when the displacement x from the equilibrium position is zero. (b) For the stretched spring $x > 0$ and $F_s < 0$ (c) For the compressed spring $x < 0$ and $F_s > 0$.(d) The plot of F_s versus x. The area of the shaded triangle represents the work done by the spring force. Due to the opposing signs of F_s and x, this work done is negative, $W_s = -kx_m^2/2$.*

The same is true when the spring is compressed with a displacement x_c (< 0). The spring force does work $W_s = -kx_c^2/2$ while the external force F does work $+ kx_c^2/2$. If the block is moved from an initial displacement x_i to a final displacement x_f, the work done by the spring force W_s is

$$W_s = -\int_{x_i}^{x_f} k\,x\,\mathrm{d}x = \frac{k\,x_i^2}{2} - \frac{k\,x_f^2}{2} \qquad (6.17)$$

Thus the work done by the spring force depends only on the end points. Specifically, if the block is pulled from x_i and allowed to return to x_i;

$$W_s = -\int_{x_i}^{x_i} k\,x\,\mathrm{d}x = \frac{k\,x_i^2}{2} - \frac{k\,x_i^2}{2}$$

$$= 0 \qquad (6.18)$$

The work done by the spring force in a cyclic process is zero. We have explicitly demonstrated that the spring force (i) is position dependent only as first stated by Hooke, ($F_s = -kx$); (ii) does work which only depends on the initial and final positions, e.g. Eq. (6.17). Thus, the spring force is a **conservative force**.

We define the potential energy $V(x)$ of the spring to be zero when block and spring system is in the equilibrium position. For an extension (or compression) x the above analysis suggests that

$$V(x) = \frac{kx^2}{2} \qquad (6.19)$$

You may easily verify that $-\mathrm{d}V/\mathrm{d}x = -k\,x$, the spring force. If the block of mass m in Fig. 6.7 is extended to x_m and released from rest, then its total mechanical energy at any arbitrary point x, where x lies between $-x_m$ and $+x_m$, will be given by

$$\frac{1}{2}k\,x_m^2 = \frac{1}{2}k\,x^2 + \frac{1}{2}m\,v^2$$

where we have invoked the conservation of mechanical energy. This suggests that the speed and the kinetic energy will be maximum at the equilibrium position, $x = 0$, i.e.,

$$\frac{1}{2}m\,v_m^2 = \frac{1}{2}k\,x_m^2$$

where v_m is the maximum speed.

or $$v_m = \sqrt{\frac{k}{m}}\,x_m$$

Note that k/m has the dimensions of [T^{-2}] and our equation is dimensionally correct. The kinetic energy gets converted to potential energy and vice versa, however, the total mechanical energy remains constant. This is graphically depicted in Fig. 6.8.

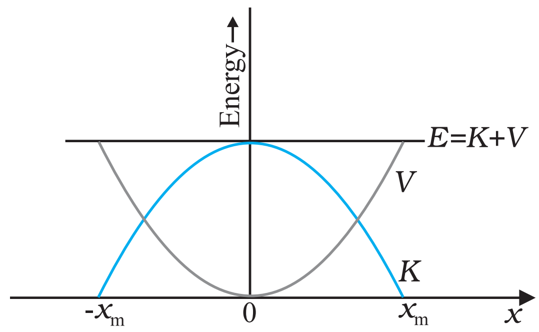


Fig. 6.8 *Parabolic plots of the potential energy V and kinetic energy K of a block attached to a spring obeying Hooke's law. The two plots are complementary, one decreasing as the other increases. The total mechanical energy E = K + V remains constant.*

▶ ***Example 6.8*** To simulate car accidents, auto manufacturers study the collisions of moving cars with mounted springs of different spring constants. Consider a typical simulation with a car of mass 1000 kg moving with a speed 18.0 km/h on a smooth road and colliding with a horizontally mounted spring of spring constant 6.25×10^3 N m^{-1}. What is the maximum compression of the spring?

Answer At maximum compression the kinetic energy of the car is converted entirely into the potential energy of the spring.

The kinetic energy of the moving car is

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2} \times 10^3 \times 5 \times 5$$

$$K = 1.25 \times 10^4 \text{ J}$$

where we have converted 18 km h^{-1} to 5 m s^{-1} **[It is useful to remember that 36 km h^{-1} = 10 m s^{-1}]**. At maximum compression x_m, the potential energy V of the spring is equal to the kinetic energy K of the moving car from the principle of conservation of mechanical energy.

$$V = \frac{1}{2}k\,x_m^2$$

$= 1.25 \times 10^4$ J

We obtain

$$x_m = 2.00 \text{ m}$$

We note that we have idealised the situation. The spring is considered to be massless. The surface has been considered to possess negligible friction. ◀

We conclude this section by making a few remarks on conservative forces.

(i) Information on time is absent from the above discussions. In the example considered above, we can calculate the compression, but not the time over which the compression occurs. A solution of Newton's Second Law for this system is required for temporal information.

(ii) Not all forces are conservative. Friction, for example, is a non-conservative force. The principle of conservation of energy will have to be modified in this case. This is illustrated in Example 6.9.

(iii) The zero of the potential energy is arbitrary. It is set according to convenience. For the spring force we took $V(x) = 0$, at $x = 0$, i.e. the unstretched spring had zero potential energy. For the constant gravitational force mg, we took $V = 0$ on the earth's surface. In a later chapter we shall see that for the force due to the universal law of gravitation, the zero is best defined at an infinite distance from the gravitational source. However, once the zero of the potential energy is fixed in a given discussion, it must be consistently adhered to throughout the discussion. You cannot change horses in midstream !

▶ ***Example 6.9*** Consider Example 6.8 taking the coefficient of friction, μ, to be 0.5 and calculate the maximum compression of the spring.

Answer In presence of friction, both the spring force and the frictional force act so as to oppose the compression of the spring as shown in Fig. 6.9.

We invoke the work-energy theorem, rather than the conservation of mechanical energy.

The change in kinetic energy is

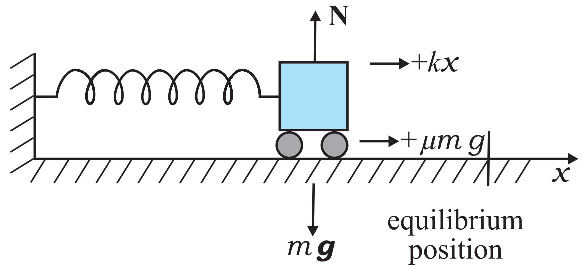


Fig. 6.9 *The forces acting on the car.*

$$\Delta K = K_f - K_i = 0 - \frac{1}{2} m v^2$$

The work done by the net force is

$$W = -\frac{1}{2} k x_m^2 - \mu m g x_m$$

Equating we have

$$\frac{1}{2} m v^2 = \frac{1}{2} k x_m^2 + \mu m g x_m$$

Now $\mu mg = 0.5 \times 10^3 \times 10 = 5 \times 10^3$ N (taking $g = 10.0$ m s^{-2}). After rearranging the above equation we obtain the following quadratic equation in the unknown x_m.

$$k x_m^2 + 2\mu m g x_m - m v^2 = 0$$

$$x_m = \frac{-\mu m g + \left[\mu^2 m^2 g^2 + m k v^2\right]^{1/2}}{k}$$

where we take the positive square root since x_m is positive. Putting in numerical values we obtain

$$x_m = 1.35 \text{ m}$$

which, as expected, is less than the result in Example 6.8.

If the two forces on the body consist of a conservative force F_c and a non-conservative force F_{nc}, the conservation of mechanical energy formula will have to be modified. By the WE theorem

$$(F_c + F_{nc})\,\Delta x = \Delta K$$

But $\quad F_c\,\Delta x = -\Delta V$

Hence, $\quad \Delta(K + V) = F_{nc}\,\Delta x$

$$\Delta E = F_{nc}\,\Delta x$$

where E is the total mechanical energy. Over the path this assumes the form

$$E_f - E_i = W_{nc}$$

where W_{nc} is the total work done by the non-conservative forces over the path. Note that