Chapter 3
Using $F = ma$

The general goal of classical mechanics is to determine what happens to a given set of objects in a given physical situation. In order to figure this out, we need to know what makes the objects move the way they do. There are two main ways of going about this task. The first one, which you are undoubtedly familiar with, involves Newton's laws. This is the subject of the present chapter. The second one, which is more advanced, is the *Lagrangian* method. This is the subject of Chapter 6. It should be noted that each of these methods is perfectly sufficient for solving any problem, and they both produce the same information in the end. But they are based on vastly different principles. We'll talk more about this in Chapter 6.

3.1 Newton's laws

In 1687 Newton published his three laws in his *Principia Mathematica*. These laws are fairly intuitive, although I suppose it's questionable to attach the adjective "intuitive" to a set of statements that weren't written down until a mere 300 years ago. At any rate, the laws may be stated as follows.

- **First law:** A body moves with constant velocity (which may be zero) unless acted on by a force.
- **Second law:** The time rate of change of the momentum of a body equals the force acting on the body.
- **Third law:** For every force on one body, there is an equal and opposite force on another body.

We could discuss for days on end the degree to which these statements are physical laws, and the degree to which they are definitions. Sir Arthur Eddington once made the unflattering remark that the first law essentially says that "every particle continues in its state of rest or uniform motion in a straight line except insofar as it doesn't." However, although the three laws might seem somewhat light on content at first glance, there's actually more to them than Eddington's comment implies. Let's look at each in turn.[1]

[1] A disclaimer: This section represents my view on which parts of the laws are definitions and which parts have content. But you should take all of this with a grain of salt. For further reading, see Anderson (1990), Keller (1987), O'Sullivan (1980), and Eisenbud (1958).

First law

One thing this law does is give a definition of zero force. Another thing it does is give a definition of an *inertial frame*, which is defined simply as a frame of reference in which the first law holds; since the term "velocity" is used, we have to state what frame we're measuring the velocity with respect to. The first law does *not* hold in an arbitrary frame. For example, it fails in the frame of a rotating turntable.[2] Intuitively, an inertial frame is one that moves with constant velocity. But this is ambiguous, because we have to say what the frame is moving with constant velocity *with respect to*. But all this aside, an inertial frame is defined as the special type of frame in which the first law holds.

So, what we now have are two intertwined definitions of "force" and "inertial frame." Not much physical content here. But the important point is that the law holds for *all* particles. So if we have a frame in which one free particle moves with constant velocity, then *all* free particles move with constant velocity. This is a statement with content. We can't have a bunch of free particles moving with constant velocity while another one is doing a fancy jig.

Second law

Momentum is defined[3] to be $m\mathbf{v}$. If m is constant,[4] then the second law says that

$$\mathbf{F} = m\mathbf{a}, \tag{3.1}$$

where $\mathbf{a} \equiv d\mathbf{v}/dt$. This law holds only in an inertial frame, which is defined by the first law.

> For things moving free or at rest,
> Observe what the first law does best.
> It defines a key frame,
> "Inertial" by name,
> Where the second law then is expressed.

You might think that the second law merely gives a definition of force, but there is more to it than that. There is a tacit implication in the law that this "force" is something that has an existence that isn't completely dependent on the particle whose "m" appears in the law (more on this in the third law below). A spring force, for example, doesn't depend at all on the particle on which it acts. And the gravitational force, GMm/r^2, depends partly on the particle and partly on something else (another mass).

[2] It's possible to modify things so that Newton's laws hold in such a frame, provided that we introduce the so-called "fictitious" forces. But we'll save this discussion for Chapter 10.

[3] We're doing everything nonrelativistically here, of course. Chapter 12 gives the relativistic modification to the $m\mathbf{v}$ expression.

[4] We'll assume in this chapter that m is constant. But don't worry, we'll get plenty of practice with changing mass (in rockets and such) in Chapter 5.

If you feel like just making up definitions, then you can define a new quantity, $\mathbf{G} = m^2\mathbf{a}$. This is a perfectly legal thing to do; you can't really go wrong in making a definition (well, unless you've already defined the quantity to be something else). However, this definition is completely useless. You can define it for every particle in the world, and for any acceleration, but the point is that the definitions don't have anything to do with each other. There is simply no (uncontrived) quantity in this world that gives accelerations in the ratio of 4 to 1 when "acting" on masses m and $2m$. The quantity $\mathbf{G}$ has nothing to do with anything except the particle you defined it for. The main thing the second law says is that there does indeed exist a quantity $\mathbf{F}$ that gives the same $m\mathbf{a}$ when acting on different particles. The statement of the existence of such a thing is far more than a definition.

Along this same line, note that the second law says that $\mathbf{F} = m\mathbf{a}$, and not, for example, $\mathbf{F} = m\mathbf{v}$, or $\mathbf{F} = m\, d^3\mathbf{x}/dt^3$. In addition to being inconsistent with the real world, these expressions are inconsistent with the first law. $\mathbf{F} = m\mathbf{v}$ would say that a nonzero velocity requires a force, in contrast with the first law. And $\mathbf{F} = md^3\mathbf{x}/dt^3$ would say that a particle moves with constant acceleration (instead of constant velocity) unless acted on by a force, also in contrast with the first law.

As with the first law, it is important to realize that the second law holds for *all* particles. In other words, if the same force (for example, the same spring stretched by the same amount) acts on two particles with masses m_1 and m_2, then Eq. (3.1) says that their accelerations are related by

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}\,. \tag{3.2}$$

This relation holds regardless of what the common force is. Therefore, once we've used one force to find the relative masses of two objects, then we know what the ratio of their a's will be when they are subjected to any other force. Of course, we haven't really defined *mass* yet. But Eq. (3.2) gives an experimental method for determining an object's mass in terms of a standard (say, 1 kg) mass. All we have to do is compare its acceleration with that of the standard mass, when acted on by the same force.

Note that $\mathbf{F} = m\mathbf{a}$ is a vector equation, so it is really three equations in one. In Cartesian coordinates, it says that $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$.

Third law

One thing this law says is that if we have two isolated particles interacting through some force, then their accelerations are opposite in direction and inversely proportional to their masses. Equivalently, the third law essentially postulates that the total momentum of an isolated system is conserved (that is, independent of time). To see this, consider two particles, each of which interacts only with the other particle and nothing else in the universe. Then we have

$$\begin{aligned}\frac{d\mathbf{p}_{\text{total}}}{dt} &= \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} \\ &= \mathbf{F}_1 + \mathbf{F}_2,\end{aligned} \tag{3.3}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are the forces acting on m_1 and m_2, respectively. This demonstrates that momentum conservation (that is, $d\mathbf{p}_{\text{total}}/dt = 0$) is equivalent to Newton's third law (that is, $\mathbf{F}_1 = -\mathbf{F}_2$). Similar reasoning holds with more than two particles, but we'll save this more general case, along with many other aspects of momentum, for Chapter 5.

There isn't much left to be defined via this law, so this is a law of pure content. It can't be a definition, anyway, because it's actually not always valid. It holds for forces of the "pushing" and "pulling" type, but it fails for the magnetic force, for example. In that case, momentum is carried off in the electromagnetic field (so the total momentum of the particles *and* the field is conserved). But we won't deal with fields here. Just particles. So the third law will always hold in any situation we'll be concerned with.

The third law contains an extremely important piece of information. It says that we will never find a particle accelerating unless there's some other particle accelerating somewhere else. The other particle might be far away, as with the earth–sun system, but it's always out there somewhere. Note that if we were given only the second law, then it would be perfectly possible for a given particle to spontaneously accelerate with nothing else happening in the universe, as long as a similar particle with twice the mass accelerated with half the acceleration when placed in the same spot, etc. This would all be fine, as far as the second law goes. We would say that a force with a certain value is acting at the point, and everything would be consistent. But the third law says that this is simply not the way the world (at least the one we live in) works. In a sense, a force without a counterpart seems somewhat like magic, whereas a force with an equal and opposite counterpart has a "cause and effect" nature, which seems (and apparently is) more physical.

In the end, however, we shouldn't attach too much significance to Newton's laws, because although they were a remarkable intellectual achievement and work spectacularly for everyday physics, they are the laws of a theory that is only approximate. Newtonian physics is a limiting case of the more correct theories of relativity and quantum mechanics, which are in turn limiting cases of yet more correct theories. The way in which particles (or waves, or strings, or whatever) interact on the most fundamental level surely doesn't bear any resemblance to what we call forces.

3.2 Free-body diagrams

The law that allows us to be quantitative is the second law. Given a force, we can apply $\mathbf{F} = m\mathbf{a}$ to find the acceleration. And knowing the acceleration, we can determine the behavior of a given object (that is, the position and velocity), provided that we are given the initial position and velocity. This process sometimes takes a bit of work, but there are two basic types of situations that commonly arise.

- In many problems, all you are given is a physical situation (for example, a block resting on a plane, strings connecting masses, etc.), and it is up to you to find all the forces acting on all the objects, using $\mathbf{F} = m\mathbf{a}$. The forces generally point in various directions, so it's easy to lose track of them. It therefore proves useful to isolate the objects and draw all the forces acting on each of them. This is the subject of the present section.
- In other problems, you are *given* the force explicitly as a function of time, position, or velocity, and the task immediately becomes the mathematical one of solving the $F = ma \equiv m\ddot{x}$ equation (we'll just deal with one dimension here). These *differential equations* can be difficult (or impossible) to solve exactly. They are the subject of Section 3.3.

Let's consider here the first of these two types of scenarios, where we are presented with a physical situation and we must determine all the forces involved. The term *free-body diagram* is used to denote a diagram with all the forces drawn on a given object. After drawing such a diagram for each object in the setup, we simply write down all the $F = ma$ equations they imply. The result will be a system of linear equations in various unknown forces and accelerations, for which we can then solve. This procedure is best understood through an example.

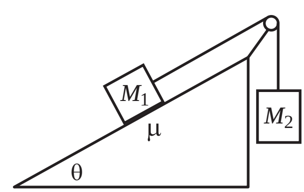

Fig. 3.1

Example (A plane and masses): Mass M_1 is held on a plane with inclination angle θ, and mass M_2 hangs over the side. The two masses are connected by a massless string which runs over a massless pulley (see Fig. 3.1). The coefficient of kinetic friction between M_1 and the plane is μ. M_1 is released from rest. Assuming that M_2 is sufficiently large so that M_1 gets pulled up the plane, what is the acceleration of the masses? What is the tension in the string?

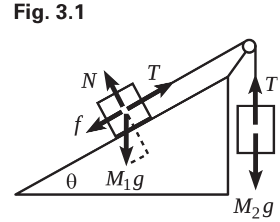

Fig. 3.2

Solution: The first thing to do is draw all the forces on the two masses. These are shown in Fig. 3.2. The forces on M_2 are gravity and the tension. The forces on M_1 are gravity, friction, the tension, and the normal force. Note that the friction force points down the plane, because we are assuming that M_1 moves up the plane.

Having drawn all the forces, we can now write down all the $F = ma$ equations. When dealing with M_1, we could break things up into horizontal and vertical components, but it is much cleaner to use the components parallel and perpendicular to the plane.[5] These two components of $\mathbf{F} = m\mathbf{a}$, along with the vertical $F = ma$ equation for M_2, give

$$\begin{aligned} T - f - M_1 g \sin\theta &= M_1 a, \\ N - M_1 g \cos\theta &= 0, \\ M_2 g - T &= M_2 a, \end{aligned} \tag{3.4}$$

where we have used the fact that the two masses accelerate at the same rate (and we have defined the positive direction for M_2 to be downward). We have also used the fact that the tension is the same at both ends of the string, because otherwise there would be a net force on some part of the string which would then undergo infinite acceleration, because it is massless.

There are four unknowns in Eq. (3.4) (namely T, a, N, and f), but only three equations. Fortunately, we have a fourth equation: $f = \mu N$, because we are assuming that M_1 is in fact moving, so we can use the expression for kinetic friction. Using this in the second equation above gives $f = \mu M_1 g \cos\theta$. The first equation then becomes $T - \mu M_1 g \cos\theta - M_1 g \sin\theta = M_1 a$. Adding this to the third equation leaves us with only a, so we find

$$a = \frac{g(M_2 - \mu M_1 \cos\theta - M_1 \sin\theta)}{M_1 + M_2} \implies T = \frac{M_1 M_2 g(1 + \mu\cos\theta + \sin\theta)}{M_1 + M_2}. \tag{3.5}$$

Note that in order for M_1 to in fact accelerate upward (that is, $a > 0$), we must have $M_2 > M_1(\mu\cos\theta + \sin\theta)$. This is clear from looking at the forces along the plane.

REMARK: If we instead assume that M_1 is sufficiently large so that it slides down the plane, then the friction force points up the plane, and we find (as you can check),

$$a = \frac{g(M_2 + \mu M_1 \cos\theta - M_1 \sin\theta)}{M_1 + M_2}, \quad \text{and} \quad T = \frac{M_1 M_2 g(1 - \mu\cos\theta + \sin\theta)}{M_1 + M_2}. \tag{3.6}$$

In order for M_1 to in fact accelerate downward (that is, $a < 0$), we must have $M_2 < M_1(\sin\theta - \mu\cos\theta)$. Therefore, the range of M_2 for which the system doesn't accelerate (that is, it just sits there, assuming that it started at rest) is

$$M_1(\sin\theta - \mu\cos\theta) \le M_2 \le M_1(\sin\theta + \mu\cos\theta). \tag{3.7}$$

If μ is very small, then M_2 must essentially be equal to $M_1 \sin\theta$ if the system is to be static. Equation (3.7) also implies that if $\tan\theta \le \mu$, then M_1 won't slide down, even if $M_2 = 0$. ♣

In problems like the one above, it's clear which things you should pick as the objects you're going to draw forces on. But in other problems, where there are

[5] When dealing with inclined planes, it's usually the case that one of these two coordinate systems works much better than the other. Sometimes it isn't clear which one, but if things get messy with one system, you can always try the other.

various different subsystems you can choose, you must be careful to include all the relevant forces on a given subsystem. Which subsystems you want to pick depends on what quantities you're trying to find. Consider the following example.

Example (Platform and pulley): A person stands on a platform-and-pulley system, as shown in Fig. 3.3. The masses of the platform, person, and pulley[6] are M, m, and μ, respectively.[7] The rope is massless. Let the person pull up on the rope so that she has acceleration a upward. (Assume that the platform is somehow constrained to stay level, perhaps by having the ends run along some rails.) Find the tension in the rope, the normal force between the person and the platform, and the tension in the rod connecting the pulley to the platform.

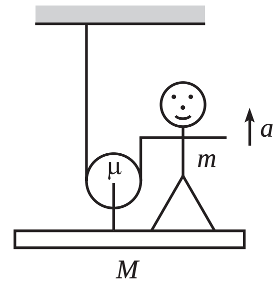

Fig. 3.3

Solution: To find the tension in the rope, we simply want to let our subsystem be the whole system (except the ceiling). If we imagine putting the system in a black box (to emphasize the fact that we don't care about any internal forces within the system), then the forces we see "protruding" from the box are the three weights (Mg, mg, and μg) downward, and the tension T upward. Applying $F = ma$ to the whole system gives

$$T - (M + m + \mu)g = (M + m + \mu)a \quad \Longrightarrow \quad T = (M + m + \mu)(g + a). \tag{3.8}$$

To find the normal force N between the person and the platform, and also the tension f in the rod connecting the pulley to the platform, it is not sufficient to consider the system as a whole. This is true because these forces are internal forces to this system, so they won't show up in any $F = ma$ equations (which involve only external forces to a system). So we must consider subsystems:

- Let's apply $F = ma$ to the person. The forces acting on the person are gravity, the normal force from the platform, and the tension from the rope (pulling downward on her hand). So we have

$$N - T - mg = ma. \tag{3.9}$$

- Now apply $F = ma$ to the platform. The forces acting on the platform are gravity, the normal force from the person, and the force upward from the rod. So we have

$$f - N - Mg = Ma. \tag{3.10}$$

- Now apply $F = ma$ to the pulley. The forces acting on the pulley are gravity, the force downward from the rod, and *twice* the tension in the rope (because it pulls

[6] Assume that the pulley's mass is concentrated at its center, so that we don't have to worry about any rotational dynamics (the subject of Chapter 8).

[7] My apologies for using μ as a mass here, since it usually denotes a coefficient of friction. Alas, there are only so many symbols for "m."

up on both sides). So we have

$$2T - f - \mu g = \mu a. \tag{3.11}$$

Note that if we add up the three previous equations, we obtain the $F = ma$ equation in Eq. (3.8), as should be the case, because the whole system is the sum of the three above subsystems. Equations (3.9)–(3.11) are three equations in the three unknowns, T, N, and f. Their sum yields the T in (3.8), and then Eqs. (3.9) and (3.11) give, respectively, as you can show,

$$N = (M + 2m + \mu)(g + a), \quad \text{and} \quad f = (2M + 2m + \mu)(g + a). \tag{3.12}$$

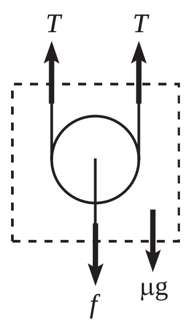

Fig. 3.4

Remarks: You can also obtain these results by considering subsystems different from the ones we chose above. For example, you can choose the pulley-plus-platform subsystem, etc. But no matter how you choose to break up the system, you will need to produce three independent $F = ma$ equations in order to solve for the three unknowns, T, N, and f.

In problems like this one, it's easy to forget to include certain forces, such as the second T in Eq. (3.11). The safest thing to do is to always isolate each subsystem, draw a box around it, and then draw all the forces that "protrude" from the box. In other words, draw the free-body diagram. Figure 3.4 shows the free-body diagram for the subsystem consisting of only the pulley. ♣

Another class of problems, similar to the above example, goes by the name of *Atwood's machines.* An Atwood's machine is the name used for any system consisting of a combination of masses, strings, and pulleys.[8] In general, the pulleys and strings can have mass, but we'll deal only with massless ones in this chapter. As we'll see in the following example, there are two basic steps in solving an Atwood's problem: (1) write down all the $F = ma$ equations, and (2) relate the accelerations of the various masses by noting that the length of the string(s) doesn't change, a fact that we call "conservation of string."

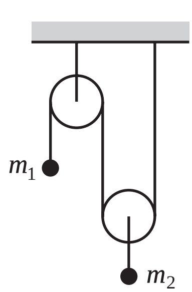

Fig. 3.5

Example (Atwood's machine): Consider the pulley system in Fig. 3.5, with masses m_1 and m_2. The strings and pulleys are massless. What are the accelerations of the masses? What is the tension in the string?

Solution: The first thing to note is that the tension T is the same everywhere throughout the massless string, because otherwise there would be an infinite acceleration of some part of the string. It then follows that the tension in the short string connected to m_2 is $2T$. This is true because there must be zero net force on the massless right pulley, because otherwise it would have infinite acceleration. The $F = ma$

[8] George Atwood (1746–1807) was a tutor at Cambridge University. He published the description of the first of his machines in Atwood (1784). For a history of Atwood's machines, see Greenslade (1985).

equations for the two masses are therefore (with upward taken to be positive)

$$T - m_1 g = m_1 a_1, \qquad 2T - m_2 g = m_2 a_2. \tag{3.13}$$

We now have two equations in the three unknowns, a_1, a_2, and T. So we need one more equation. This is the "conservation of string" fact, which relates a_1 and a_2. If we imagine moving m_2 and the right pulley up a distance d, then a length $2d$ of string has disappeared from the two parts of the string touching the right pulley. This string has to go somewhere, so it ends up in the part of the string touching m_1 (see Fig. 3.6). Therefore, m_1 goes down by a distance $2d$. In other words, $y_1 = -2y_2$, where y_1 and y_2 are measured relative to the initial locations of the masses. Taking two time derivatives of this statement gives our desired relation between a_1 and a_2,

$$a_1 = -2a_2. \tag{3.14}$$

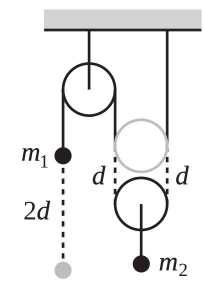

Fig. 3.6

Combining this with Eq. (3.13), we can now solve for a_1, a_2, and T. The result is

$$a_1 = g\frac{2m_2 - 4m_1}{4m_1 + m_2}, \quad a_2 = g\frac{2m_1 - m_2}{4m_1 + m_2}, \quad T = \frac{3m_1 m_2 g}{4m_1 + m_2}. \tag{3.15}$$

REMARKS: There are all sorts of limits and special cases that we can check here. A couple are: (1) If $m_2 = 2m_1$, then Eq. (3.15) gives $a_1 = a_2 = 0$, and $T = m_1 g$. Everything is at rest. (2) If $m_2 \gg m_1$, then Eq. (3.15) gives $a_1 = 2g$, $a_2 = -g$, and $T = 3m_1 g$. In this case, m_2 is essentially in free fall, while m_1 gets yanked up with acceleration $2g$. The value of T is exactly what is needed to make the net force on m_1 equal to $m_1(2g)$, because $T - m_1 g = 3m_1 g - m_1 g = m_1(2g)$. You can check the case where $m_1 \gg m_2$.

For the more general case where there are N masses instead of two, the "conservation of string" statement is a single equation that relates all N accelerations. It is most easily obtained by imagining moving $N-1$ of the masses, each by an arbitrary amount, and then seeing what happens to the last mass. Note that these arbitrary motions undoubtedly do *not* correspond to the actual motions of the masses. This is fine; the single "conservation of string" equation has nothing to do with the N $F = ma$ equations. The combination of all $N+1$ equations is needed to constrain the motions down to a unique set. ♣

In the problems and exercises for this chapter, you will encounter some strange Atwood's setups. But no matter how complicated they get, there are only two things you need to do to solve them, as mentioned above: write down the $F = ma$ equations for all the masses (which may involve relating the tensions in various strings), and then relate the accelerations of the masses, using "conservation of string."

It may seem, with the angst it can bring,
That an Atwood's machine's a cruel thing.
But you just need to say
That F is ma,
And use conservation of string!

3.3 Solving differential equations

Let's now consider the type of problem where we are *given* the force as a function of time, position, or velocity, and our task is to solve the $F = ma \equiv m\ddot{x}$ differential equation to find the position, $x(t)$, as a function of time.[9] In what follows, we will develop a few techniques for solving differential equations. The ability to apply these techniques dramatically increases the number of systems we can understand.

It's also possible for the force F to be a function of higher derivatives of x, in addition to the quantities t, x, and $v \equiv \dot{x}$. But these cases don't arise much, so we won't worry about them. The $F = ma$ differential equation we want to solve is therefore (we'll just work in one dimension here)

$$m\ddot{x} = F(t, x, v). \tag{3.16}$$

In general, this equation cannot be solved exactly for $x(t)$.[10] But for most of the problems we'll deal with, it can be solved. The problems we'll encounter will often fall into one of three special cases, namely, where F is a function of t only, or x only, or v only. In all of these cases, we must invoke the given initial conditions, $x_0 \equiv x(t_0)$ and $v_0 \equiv v(t_0)$, to obtain our final solutions. These initial conditions will appear in the limits of the integrals in the following discussion.[11]

Note: You may want to just skim the following page and a half, and then refer back as needed. Don't try to memorize all the different steps. We present them only for completeness. The whole point here can basically be summarized by saying that sometimes you want to write $\ddot{x}$ as dv/dt, and sometimes you want to write it as $v\,dv/dx$ (see Eq. (3.20)). Then you "simply" have to separate variables and integrate. We'll go through the three special cases, and then we'll do some examples.

- *F is a function of t only: $F = F(t)$.*
  Since $a = d^2x/dt^2$, we just need to integrate $F = ma$ twice to obtain $x(t)$. Let's do this in a very systematic way, to get used to the general procedure. First, write $F = ma$ as

$$m\frac{dv}{dt} = F(t). \tag{3.17}$$

[9] In some setups, such as in Problem 3.11, the force isn't given, so you have to figure out what it is. But the main part of the problem is still solving the resulting differential equation.

[10] You can always solve for $x(t)$ *numerically*, to any desired accuracy. This topic is discussed in Section 1.4.

[11] It is no coincidence that we need *two* initial conditions to completely specify the solution to our *second*-order (meaning the highest derivative of x that appears is the second one) $F = m\ddot{x}$ differential equation. It is a general result (which we'll just accept here) that the solution to an nth-order differential equation has n free parameters, which are determined by the initial conditions.

Then separate variables and integrate both sides to obtain[12]

$$m\int_{v_0}^{v(t)} dv' = \int_{t_0}^{t} F(t')\,dt'. \tag{3.18}$$

We have put primes on the integration variables so that we don't confuse them with the limits of integration, but in practice we usually don't bother with them. The integral of dv' is just v', so Eq. (3.18) yields v as a function of t, that is, $v(t)$. We can then separate variables in $dx/dt \equiv v(t)$ and integrate to obtain

$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\,dt'. \tag{3.19}$$

This yields x as a function of t, that is, $x(t)$. This procedure might seem like a cumbersome way to simply integrate something twice. That's because it is. But the technique proves more useful in the following case.

- *F is a function of x only: $F = F(x)$.*
  We will use

$$a = \frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \tag{3.20}$$

to write $F = ma$ as

$$mv\frac{dv}{dx} = F(x). \tag{3.21}$$

Now separate variables and integrate both sides to obtain

$$m\int_{v_0}^{v(x)} v'\,dv' = \int_{x_0}^{x} F(x')\,dx'. \tag{3.22}$$

The integral of v' is $v'^2/2$, so the left-hand side involves the square of $v(x)$. Taking the square root, this gives v as a function of x, that is, $v(x)$. Separating variables in $dx/dt \equiv v(x)$ then yields

$$\int_{x_0}^{x(t)} \frac{dx'}{v(x')} = \int_{t_0}^{t} dt'. \tag{3.23}$$

Assuming that we can do the integral on the left-hand side, this equation gives t as a function of x. We can then (in principle) invert the result to obtain x as a function of t, that is, $x(t)$. The unfortunate thing about this case is that the integral in Eq. (3.23) might not be doable. And even if it is, it might not be possible to invert $t(x)$ to produce $x(t)$.

[12] If you haven't seen such a thing before, the act of multiplying both sides by the infinitesimal quantity dt might make you feel a bit uneasy. But it is in fact quite legal. If you wish, you can imagine working with the small (but not infinitesimal) quantities Δv and Δt, for which it is certainly legal to multiply both sides by Δt. Then you can take a discrete sum over many Δt intervals, and then finally take the limit $\Delta t \to 0$, which results in the integral in Eq. (3.18).

- *F is a function of v only: $F = F(v)$.*
  Write $F = ma$ as

$$m\frac{dv}{dt} = F(v). \tag{3.24}$$

  Separate variables and integrate both sides to obtain

$$m\int_{v_0}^{v(t)} \frac{dv'}{F(v')} = \int_{t_0}^{t} dt'. \tag{3.25}$$

  Assuming that we can do this integral, it yields t as a function of v, and hence (in principle) v as a function of t, that is, $v(t)$. We can then integrate $dx/dt \equiv v(t)$ to obtain $x(t)$ from

$$\int_{x_0}^{x(t)} dx' = \int_{t_0}^{t} v(t')\, dt'. \tag{3.26}$$

  Note: In this $F = F(v)$ case, if we want to find v as a function of x, $v(x)$, then we should write a as $v(dv/dx)$ and integrate

$$m\int_{v_0}^{v(x)} \frac{v'\, dv'}{F(v')} = \int_{x_0}^{x} dx'. \tag{3.27}$$

  We can then obtain $x(t)$ from Eq. (3.23), if desired.

When dealing with the initial conditions, we have chosen to put them in the limits of integration above. If you wish, you can perform the integrals without any limits, and just tack on a constant of integration to your result. The constant is then determined by the initial conditions.

Again, as mentioned above, you do *not* have to memorize the above three procedures, because there are variations, depending on what you're given and what you want to solve for. All you have to remember is that $\ddot{x}$ can be written as either dv/dt or $v\, dv/dx$. One of these will get the job done (namely, the one that makes only two of the three variables, t, x, and v, appear in your differential equation). And then be prepared to separate variables and integrate as many times as needed.[13]

> a is dv by dt.
> Is this useful? There's no guarantee.
> If it leads to "Oh, heck!" s,
> Take dv by dx,
> And then write down its product with v.

[13] We want only two of the variables to appear in the differential equation because the goal is to separate variables and integrate, and because equations have only two sides. If equations were triangles, it would be a different story.

Example (Gravitational force): A particle of mass m is subject to a constant force $F = -mg$. The particle starts at rest at height h. Because this constant force falls into all of the above three categories, we should be able to solve for $y(t)$ in two ways:

(a) Find $y(t)$ by writing a as dv/dt.
(b) Find $y(t)$ by writing a as $v\,dv/dy$.

Solution:

(a) $F = ma$ gives $dv/dt = -g$. Multiplying by dt and integrating yields $v = -gt + A$, where A is a constant of integration.[14] The initial condition $v(0) = 0$ gives $A = 0$. Therefore, $dy/dt = -gt$. Multiplying by dt and integrating yields $y = -gt^2/2 + B$. The initial condition $y(0) = h$ gives $B = h$. Therefore,

$$y = h - \frac{1}{2}gt^2. \tag{3.28}$$

(b) $F = ma$ gives $v\,dv/dy = -g$. Separating variables and integrating yields $v^2/2 = -gy + C$. The initial condition $v(h) = 0$ gives $v^2/2 = -gy + gh$. Therefore, $v \equiv dy/dt = -\sqrt{2g(h-y)}$. We have chosen the negative square root because the particle is falling. Separating variables gives

$$\int \frac{dy}{\sqrt{h-y}} = -\sqrt{2g}\int dt. \tag{3.29}$$

This yields $2\sqrt{h-y} = \sqrt{2g}\,t$, where we have used the initial condition $y(0) = h$. Hence, $y = h - gt^2/2$, as in part (a). In part (b) here, we essentially derived conservation of energy, as we'll see in Chapter 5.

Example (Dropped ball): A beach ball is dropped from rest at height h. Assume that the drag force[15] from the air takes the form of $F_{\text{d}} = -\beta v$. Find the velocity and height as a function of time.

Solution: For simplicity in future formulas, let's write the drag force as $F_{\text{d}} = -\beta v \equiv -m\alpha v$ (otherwise we'd have a bunch of $1/m$'s floating around). Taking upward to be the positive y direction, the force on the ball is

$$F = -mg - m\alpha v. \tag{3.30}$$

[14] We'll do this example by adding on constants of integration which are then determined by the initial conditions. We'll do the following example by putting the initial conditions in the limits of integration.

[15] The drag force is roughly proportional to v as long as the speed is fairly small (say, less than 10 m/s). For large speeds (say, greater than 100 m/s), the drag force is roughly proportional to v^2. But these approximate cutoffs depend on various things, and in any event there is a messy transition region between the two cases.

Note that v is negative here, because the ball is falling, so the drag force points upward, as it should. Writing $F = m\,dv/dt$ and separating variables gives

$$\int_0^{v(t)} \frac{dv'}{g+\alpha v'} = -\int_0^t dt'. \tag{3.31}$$

Integration yields $\ln(1+\alpha v/g) = -\alpha t$. Exponentiation then gives

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right). \tag{3.32}$$

Writing $dy/dt \equiv v(t)$, and then separating variables and integrating to obtain $y(t)$, yields

$$\int_h^{y(t)} dy' = -\frac{g}{\alpha}\int_0^t \left(1 - e^{-\alpha t'}\right) dt'. \tag{3.33}$$

Therefore,

$$y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right). \tag{3.34}$$

REMARKS:

1. Let's look at some limiting cases. If t is very small (more precisely, if $\alpha t \ll 1$), then we can use $e^{-x} \approx 1 - x + x^2/2$ to make approximations to leading order in t. You can show that Eq. (3.32) gives $v(t) \approx -gt$. This makes sense, because the drag force is negligible at the start, so the ball is essentially in freefall. And likewise you can show that Eq. (3.34) gives $y(t) \approx h - gt^2/2$, which is again the freefall result.

   We can also look at large t. In this case, $e^{-\alpha t}$ is essentially equal to zero, so Eq. (3.32) gives $v(t) \approx -g/\alpha$. (This is the "terminal velocity." Its value makes sense, because it is the velocity for which the total force, $-mg - m\alpha v$, vanishes.) And Eq. (3.34) gives $y(t) \approx h - (g/\alpha)t + g/\alpha^2$. Interestingly, we see that for large t, g/α^2 is the distance our ball lags behind another ball that started out already at the terminal velocity, $-g/\alpha$.
2. You might think that the velocity in Eq. (3.32) doesn't depend on m, since no m's appear. However, there is an m hidden in α. The quantity α (which we introduced just to make our formulas look a little nicer) was defined by $F_\text{d} = -\beta v \equiv -m\alpha v$. But the quantity $\beta \equiv m\alpha$ is roughly proportional to the cross-sectional area, A, of the ball. Therefore, $\alpha \propto A/m$. Two balls of the same size, one made of lead and one made of styrofoam, have the same A but different m's. So their α's are different, and they fall at different rates.

   If we have a solid ball with density ρ and radius r, then $\alpha \propto A/m \propto r^2/(\rho r^3) = 1/\rho r$. For large dense objects in a thin medium such as air, the quantity α is small, so the drag effects are not very noticeable over short times (because if we include the next term in the expansion for v, we obtain $v(t) \approx -gt + \alpha g t^2/2$). Large dense objects therefore all fall at roughly the same rate, with an acceleration essentially equal to g. But if the air were much thicker, then all the α's would be larger, and maybe it would have taken Galileo a bit longer to come to his conclusions.

   What would you have thought, Galileo,
   If instead you dropped cows and did say, "Oh!

To lessen the sound
Of the moos from the ground,
They should fall not through air, but through mayo!"[16] ♣

3.4 Projectile motion

Consider a ball thrown through the air, not necessarily vertically. We will neglect air resistance in the following discussion. Things get a bit more complicated when this is included, as Exercise 3.53 demonstrates.

Let x and y be the horizontal and vertical positions, respectively. The force in the x direction is $F_x = 0$, and the force in the y direction is $F_y = -mg$. So $\mathbf{F} = m\mathbf{a}$ gives

$$\ddot{x} = 0, \quad \text{and} \quad \ddot{y} = -g. \tag{3.35}$$

Note that these two equations are "decoupled." That is, there is no mention of y in the equation for $\ddot{x}$, and vice versa. The motions in the x and y directions are therefore completely independent. The classic demonstration of the independence of the x and y motions is the following. Fire a bullet horizontally (or, preferably, just imagine firing a bullet horizontally), and at the same time drop a bullet from the height of the gun. Which bullet will hit the ground first? (Neglect air resistance, the curvature of the earth, etc.) The answer is that they will hit the ground at the same time, because the effect of gravity on the two y motions is exactly the same, independent of what is going on in the x direction.

If the initial position and velocity are (X, Y) and (V_x, V_y), then we can easily integrate Eq. (3.35) to obtain

$$\dot{x}(t) = V_x, \quad \text{and} \quad \dot{y}(t) = V_y - gt. \tag{3.36}$$

Integrating again gives

$$x(t) = X + V_x t, \quad \text{and} \quad y(t) = Y + V_y t - \frac{1}{2}gt^2. \tag{3.37}$$

These equations for the speeds and positions are all you need to solve a projectile problem.

[16] It's actually much more likely that Galileo obtained his "all objects fall at the same rate in a vacuum" result by rolling balls down planes than by dropping balls from the Tower of Pisa; see Adler and Coulter (1978). So I suppose this limerick is relevant only in the approximation of the proverbial spherical cow.

Example (Throwing a ball):

(a) For a given initial speed, at what inclination angle should a ball be thrown so that it travels the maximum horizontal distance by the time it returns to the ground? Assume that the ground is horizontal, and that the ball is released from ground level.

(b) What is the optimal angle if the ground is sloped upward at an angle β (or downward, if β is negative)?

Solution:

(a) Let the inclination angle be θ, and let the initial speed be V. Then the horizontal speed is always $V_x = V\cos\theta$, and the initial vertical speed is $V_y = V\sin\theta$. The first thing we need to do is find the time t in the air. We know that the vertical speed is zero at time $t/2$, because the ball is moving horizontally at the highest point. So the second of Eqs. (3.36) gives $V_y = g(t/2)$. Therefore, $t = 2V_y/g$.[17] The first of Eqs. (3.37) tells us that the horizontal distance traveled is $d = V_x t$. Using $t = 2V_y/g$ in this gives

$$d = \frac{2V_xV_y}{g} = \frac{V^2(2\sin\theta\cos\theta)}{g} = \frac{V^2\sin 2\theta}{g}. \tag{3.38}$$

The $\sin 2\theta$ factor has a maximum at $\theta = \pi/4$. The maximum horizontal distance traveled is then $d_{\text{max}} = V^2/g$.

REMARK: For $\theta = \pi/4$, you can show that the maximum height achieved is $V^2/4g$. This is half the maximum height of $V^2/2g$ (as you can show) if the ball is thrown straight up. Note that any possible distance you might want to find in this problem must be proportional to V^2/g, by dimensional analysis. The only question is what the numerical factor is. ♣

(b) As in part (a), the first thing we need to do is find the time t in the air. If the ground is sloped at an angle β, then the equation for the line of the ground is $y = (\tan\beta)x$. The path of the ball is given in terms of t by

$$x = (V\cos\theta)t, \quad \text{and} \quad y = (V\sin\theta)t - \frac{1}{2}gt^2, \tag{3.39}$$

where θ is the angle of the throw, as measured with respect to the horizontal (not the ground). We must solve for the t that makes $y = (\tan\beta)x$, because this gives the time when the path of the ball intersects the line of the ground. Using Eq. (3.39), we find that $y = (\tan\beta)x$ when

$$t = \frac{2V}{g}(\sin\theta - \tan\beta\cos\theta). \tag{3.40}$$

[17] Alternatively, the time of flight can be found from the second of Eqs. (3.37), which says that the ball returns to the ground when $V_y t = gt^2/2$. We will have to use this second strategy in part (b), where the trajectory is not symmetric around the maximum.

(There is, of course, also the solution $t = 0$.) Plugging this into the expression for x in Eq. (3.39) gives

$$x = \frac{2V^2}{g}(\sin\theta\cos\theta - \tan\beta\cos^2\theta). \tag{3.41}$$

We must now maximize this value for x, which is equivalent to maximizing the distance along the slope. Setting the derivative with respect to θ equal to zero, and using the double-angle formulas, $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$, we find $\tan\beta = -\cot 2\theta$. This can be rewritten as $\tan\beta = -\tan(\pi/2 - 2\theta)$. Therefore, $\beta = -(\pi/2 - 2\theta)$, so we have

$$\theta = \frac{1}{2}\left(\beta + \frac{\pi}{2}\right). \tag{3.42}$$

In other words, the throwing angle should bisect the angle between the ground and the vertical.

REMARKS:

1. For $\beta \approx \pi/2$, we have $\theta \approx \pi/2$, as should be the case. For $\beta = 0$, we have $\theta = \pi/4$, as we found in part (a). And for $\beta \approx -\pi/2$, we have $\theta \approx 0$, which makes sense.
2. A quicker method of obtaining the time in Eq. (3.40) is the following. Consider the set of tilted axes parallel and perpendicular to the ground; let these be the x' and y' axes, respectively. The initial velocity in the y' direction is $V\sin(\theta-\beta)$, and the acceleration in this direction is $g\cos\beta$. The time in the air is twice the time it takes the ball to reach the maximum "height" above the ground (measured in the y' direction), which occurs when the velocity in the y' direction is instantaneously zero. The total time is therefore $2V\sin(\theta - \beta)/(g\cos\beta)$, which you can show is equivalent to the time in Eq. (3.40). Note that the $g\sin\beta$ acceleration in the x' direction is irrelevant in calculating this time. In the present example, using these tilted axes doesn't save a huge amount of time, but in some situations (see Exercise 3.50) the tilted axes can save you a lot of grief.
3. An interesting fact about the motion of the ball in the maximum-distance case is that the initial and final velocities are perpendicular to each other. The demonstration of this is the task of Problem 3.16.
4. Substituting the value of θ from Eq. (3.42) into Eq. (3.41), you can show (after a bit of algebra) that the maximum distance traveled along the tilted ground is

$$d = \frac{x}{\cos\beta} = \frac{V^2/g}{1 + \sin\beta}. \tag{3.43}$$

Solving for V, we have $V^2 = g(d + d\sin\beta)$. This can be interpreted as saying that the minimum speed at which a ball must be thrown in order to pass over a wall of height h, at a distance L away on level ground, is given by $V^2 = g(\sqrt{L^2 + h^2} + h)$. This checks in the limits of $h \to 0$ and $L \to 0$.
5. A compilation of many other projectile results can be found in Buckmaster (1985). ♣

Along with the bullet example mentioned above, another classic example of the independence of the x and y motions is the "hunter and monkey" problem. In it, a hunter aims an arrow (a toy one, of course) at a monkey hanging from a branch in a tree. The monkey, thinking he's being clever, tries to avoid the arrow by letting go of the branch right when he sees the arrow released. The unfortunate consequence of this action is that he in fact *will* get hit, because gravity acts on both him and the arrow in the same way; they both fall the same distance relative to where they would have been if there were no gravity. And the monkey *would* get hit in such a case, because the arrow is initially aimed at him. You can work this out in Exercise 3.44, in a more peaceful setting involving fruit.

If a monkey lets go of a tree,
The arrow will hit him, you see,
Because both heights are pared
By a half gt^2
From what they would be with no g.

3.5 Motion in a plane, polar coordinates

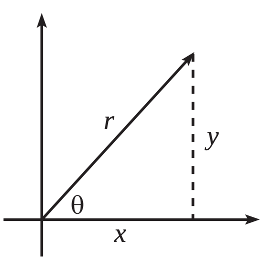

Fig. 3.7

When dealing with problems where the motion lies in a plane, it is often convenient to work with polar coordinates, r and θ. These are related to the Cartesian coordinates by (see Fig. 3.7)

$$x = r\cos\theta, \quad \text{and} \quad y = r\sin\theta. \tag{3.44}$$

Depending on the problem, either Cartesian or polar coordinates are easier to use. It is usually clear from the setup which is better. For example, if the problem involves circular motion, then polar coordinates are a good bet. But to use polar coordinates, we need to know what Newton's second law looks like when written in terms of them. Therefore, the goal of the present section is to determine what $\mathbf{F} = m\mathbf{a} \equiv m\ddot{\mathbf{r}}$ looks like when written in terms of polar coordinates.

At a given position $\mathbf{r}$ in the plane, the basis vectors in polar coordinates are $\hat{\mathbf{r}}$, which is a unit vector pointing in the radial direction; and $\hat{\boldsymbol{\theta}}$, which is a unit vector pointing in the counterclockwise tangential direction. In polar coordinates, a general vector may be written as

$$\mathbf{r} = r\hat{\mathbf{r}}. \tag{3.45}$$

Since the goal of this section is to find $\ddot{\mathbf{r}}$, we must, in view of Eq. (3.45), get a handle on the time derivative of $\hat{\mathbf{r}}$. And we'll eventually need the derivative of $\hat{\boldsymbol{\theta}}$, too. In contrast with the fixed Cartesian basis vectors ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$), the polar basis vectors ($\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$) change as a point moves around in the plane. We can find $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\boldsymbol{\theta}}}$ in the following way. In terms of the Cartesian basis, Fig. 3.8 shows that

$$\begin{aligned}\hat{\mathbf{r}} &= \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}},\\ \hat{\boldsymbol{\theta}} &= -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}.\end{aligned} \tag{3.46}$$

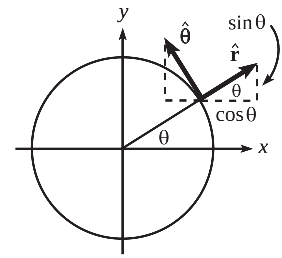

Fig. 3.8

Taking the time derivative of these equations gives

$$\begin{aligned}\dot{\hat{\mathbf{r}}} &= -\sin\theta\,\dot{\theta}\hat{\mathbf{x}} + \cos\theta\,\dot{\theta}\hat{\mathbf{y}},\\ \dot{\hat{\boldsymbol{\theta}}} &= -\cos\theta\,\dot{\theta}\hat{\mathbf{x}} - \sin\theta\,\dot{\theta}\hat{\mathbf{y}}.\end{aligned} \tag{3.47}$$

Using Eq. (3.46), we arrive at the nice clean expressions,

$$\dot{\hat{\mathbf{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}, \quad \text{and} \quad \dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\mathbf{r}}. \tag{3.48}$$

These relations are fairly evident if you look at what happens to the basis vectors as $\mathbf{r}$ moves a tiny distance in the tangential direction. Note that the basis vectors do not change as $\mathbf{r}$ moves in the radial direction. We can now start differentiating Eq. (3.45). One derivative gives (yes, the product rule works fine here)

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{\mathbf{r}}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}. \tag{3.49}$$

This makes sense, because $\dot{r}$ is the velocity in the radial direction, and $r\dot{\theta}$ is the velocity in the tangential direction, often written as $r\omega$ (where $\omega \equiv \dot{\theta}$ is the angular velocity, or "angular frequency").[18] Differentiating Eq. (3.49) then gives

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{\mathbf{r}}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}}\\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}(\dot{\theta}\hat{\boldsymbol{\theta}}) + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}(-\dot{\theta}\hat{\mathbf{r}})\\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}.\end{aligned} \tag{3.50}$$

Finally, equating $m\ddot{\mathbf{r}}$ with $\mathbf{F} \equiv F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$ gives the radial and tangential forces as

$$\begin{aligned}F_r &= m(\ddot{r} - r\dot{\theta}^2),\\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta}).\end{aligned} \tag{3.51}$$

(See Exercise 3.67 for a slightly different derivation of these equations.) Let's look at each of the four terms on the right-hand sides of Eqs. (3.51).

[18] For $r\dot{\theta}$ to be the tangential velocity, we must measure θ in radians and not degrees. Then $r\theta$ is by definition the position along the circumference, so $r\dot{\theta}$ is the velocity along the circumference.

- The $m\ddot{r}$ term is quite intuitive. For radial motion, it simply states that $F = ma$ along the radial direction.
- The $mr\ddot{\theta}$ term is also quite intuitive. For circular motion, it states that $F = ma$ along the tangential direction, because $r\ddot{\theta}$ is the second derivative of the distance $r\theta$ along the circumference.
- The $-mr\dot{\theta}^2$ term is also fairly clear. For circular motion, it says that the radial force is $-m(r\dot{\theta})^2/r = -mv^2/r$, which is the familiar force that causes the centripetal acceleration, v^2/r. See Problem 3.20 for an alternate (and quicker) derivation of this v^2/r result.
- The $2m\dot{r}\dot{\theta}$ term isn't so obvious. It is associated with the *Coriolis* force. There are various ways to look at this term. One is that it exists in order to keep angular momentum conserved. We'll have a great deal to say about the Coriolis force in Chapter 10.

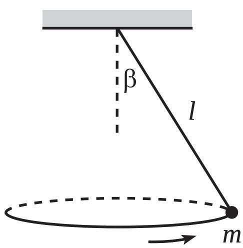

Fig. 3.9

Example (Circular pendulum): A mass hangs from a massless string of length ℓ. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making a constant angle β with the vertical (see Fig. 3.9). What is the angular frequency, ω, of this motion?

Solution: The mass travels in a circle, so the horizontal radial force must be $F_r = mr\dot{\theta}^2 \equiv mr\omega^2$ (with $r = \ell \sin\beta$), directed radially inward. The forces on the mass are the tension in the string, T, and gravity, mg (see Fig. 3.10). There is no acceleration in the vertical direction, so $F = ma$ in the vertical and radial directions gives, respectively,

$$\begin{aligned} T\cos\beta - mg &= 0, \\ T\sin\beta &= m(\ell\sin\beta)\omega^2. \end{aligned} \tag{3.52}$$

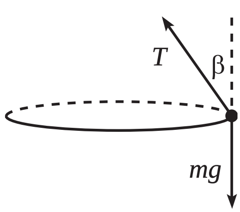

Fig. 3.10

Solving for ω gives

$$\omega = \sqrt{\frac{g}{\ell\cos\beta}}\,. \tag{3.53}$$

If $\beta \approx 90°$, then $\omega \to \infty$, which makes sense. And if $\beta \approx 0$, then $\omega \approx \sqrt{g/\ell}$, which happens to equal the frequency of a plane pendulum of length ℓ. The task of Exercise 3.60 is to explain why.

3.6 Problems

Section 3.2: Free-body diagrams

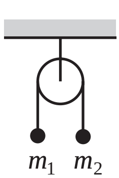

Fig. 3.11

3.1. **Atwood's machine** *

A massless pulley hangs from a fixed support. A massless string connecting two masses, m_1 and m_2, hangs over the pulley (see Fig. 3.11). Find the acceleration of the masses and the tension in the string.