Q. A circle C_1 passes through the origin and has its centre on $y = x$. If it intersects another circle C_2: $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally at points P and Q, which of the following statements is(are) correct?

[A] The circle C_1 does **NOT** pass through (2,2)

[B] Equation of common chord of C_2 and C_1 is $x + 2y = 5$

[C] Equation of a circle passing through points P and Q and having its centre on x axis is

$$x^2 + y^2 - x - 5 = 0$$

[D] Equation of a circle passing through points P, Q and $(1,1)$ is

$$(x - 1)(x - 2) + (y - 1)(y - 3) = 0$$

Answer: [B], [C], [D]

Solution:

Let

$$C_1: x^2 + y^2 + 2gx + 2fy + c = 0$$

Given

(i) C_1 passes through (0,0) implies $c = 0$

(ii) Centre $(-g, -f)$ passes through $y = x$ which implies $g = f$

(iii) C_2 and C_1 are orthogonal implies

$$2g(-2) + 2f(-3) = c + 10$$

Solving all 3 equations we get $g = f = -1,\ c = 0$. Hence

$$C_1: x^2 + y^2 - 2x - 2y = 0$$

Option [A] is wrong since C_1 satisfies (2,2)

Option [B] is correct since eqn of common chord is $L: C_1 - C_2 = 0$ ie

$$(x^2 + y^2 - 2x - 2y) - (x^2 + y^2 - 4x - 6y + 10) = 0$$

$$2x + 4y - 10 = 0 \text{ or } x + 2y = 5$$

Option [C] is correct since a circle passing through point of intersections of 2 circles is given by

$$C_3 : S + \lambda L = 0$$

$$x^2 + y^2 - 2x - 2y + \lambda(x + 2y - 5) = 0$$

$$x^2 + y^2 + (\lambda - 2)x + (2\lambda - 2)y - 5\lambda = 0$$

The centre of this circle is $(\frac{2-\lambda}{2}, 1 - \lambda)$. Since centre lies on x axis, its y coordinate must be zero

$$1 - \lambda = 0 \text{ or } \lambda = 1$$

The eqn of the circle on putting $\lambda = 1$ is

$$x^2 + y^2 - x - 5 = 0$$

Similarly **Option [D]** is also correct

This time the eqn $x^2 + y^2 + (\lambda - 2)x + (2\lambda - 2)y - 5\lambda = 0$ passes through (1,1)

ie

$$1 + 1 + (\lambda - 2) + (2\lambda - 2) - 5\lambda = 0 \implies \lambda = -1$$

The eqn of the circle on putting $\lambda = -1$ is

$$x^2 + y^2 - 3x - 4y + 5 = 0$$

Alternate method for Option [D]

Let O_1 and O_2 be the centres C_2 and C_1 respectively

Things to observe:

(i) Circle passes through $(1,1)$ ie O_1 **(centre of $\boldsymbol{C_1}$)**

(ii) $\angle O_1PO_2 = 90° = \angle O_1QO_2$ since C_2 and C_1 intersect orthogonally at P and Q

Solution: The circle passing through P , Q and O_1 implies it must also pass through O_2 since angle in a semicircle is 90 degrees. Further O_1O_2 is a diameter of the circle, hence its equation in diametric form is

$$(x-1)(x-2)+(y-1)(y-3)=0$$