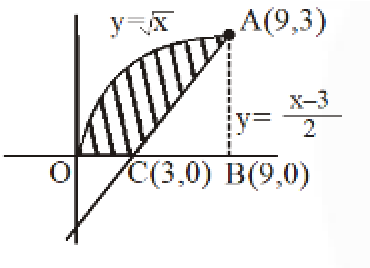
Area Under the Curve :

1. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x - 3 = 0$, x-axis, and lying in the first quadrant is (JEE Main 2013)

   a. 36 b. 18

   c. 27 /4 d. 9

Sol: Since both the curves intersect at point A



$\Rightarrow \sqrt{x} = (x - 3) / 2$

$\Rightarrow \sqrt{x} - 2x - 3 = 0$

$\Rightarrow \sqrt{x} = 3, -1 \Rightarrow x = 9$

Required area ⇒

$\int_0^9 \sqrt{x}\, dx$ - area of $\Delta ABC \Rightarrow 2\sqrt{x^3}|_0^3$ - $0.5 * 6 * 3$

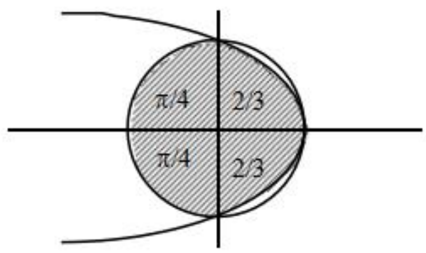
$\Rightarrow 18 - 9 = 9$

2. The area of the region described by $A = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } y^2 \leq 1 - x\}$ (JEE Main 2014)

   a. $\frac{\pi}{2}+\frac{4}{3}$ b. $\frac{\pi}{2}-\frac{4}{3}$

   c. $\frac{\pi}{2}-\frac{2}{3}$ d. $\frac{\pi}{2}+\frac{2}{3}$

Sol: The area of the shaded region



$\Rightarrow \frac{\pi}{2} + 2\int_{y=0}^{1} x\, dy$

$\Rightarrow \frac{\pi}{2} + 2\int_0^1 1 - y^2\, dy$

$\Rightarrow \frac{\pi}{2} + 2(1 - 1/3)$

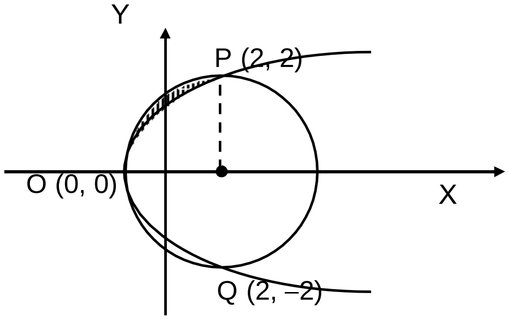
$\Rightarrow \frac{\pi}{2} + 4/3$

3. The area (in sq. units) of the region $\{(x, y) : x^2 + y^2 \leq 4x \text{ and } y^2 \geq 2x, x \geq 0, y \geq 0\}$ (JEE Main 2016)

   a. $\pi - \frac{8}{3}$ b. $\pi - \frac{4\sqrt{2}}{3}$

   c. $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ d. $\pi - \frac{4}{3}$

Sol: The point of intersection of the curve $x^2 + y^2 = 4x$ and $y^2 = 2x$ are (0, 0) and (2, 2) for $x \geq 0$ and $y \geq 0$



So, the required area is

$\Rightarrow \frac{\pi}{4} * 4 - \int_0^2 \sqrt{2x}\, dx$

$\Rightarrow \pi - \sqrt{2} * 2/3 * (\sqrt{x^3})|_0^2$

$\Rightarrow \pi - \frac{8}{3}$

4. The area (in sq. units) of the region {(x, y) : x ≥ 0, x + y ≤ 3, x^2 ≤ 4y and y ≤ 1 + √x} is (JEE Main 2017)

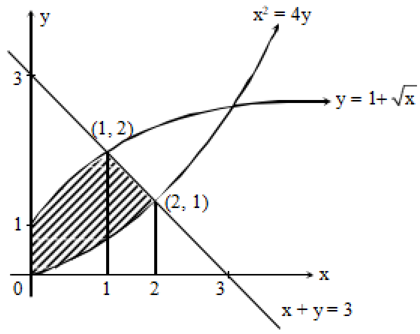
a. $\frac{59}{12}$ b. $\frac{3}{2}$

c. $\frac{7}{3}$ d. $\frac{5}{2}$

Sol: The area of the shaded region in the figure is:

$$\Rightarrow \int_0^1 1+\sqrt{x}\,dx + \int_1^2 (3-x)dx - \int_0^2 \frac{x^2}{4}dx$$

on solving ⇒ $\frac{5}{2}$ sq. units



5. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$, and α, β (α < β) be the roots of the quadratic equation

$18x^2+9\pi x+\pi^2 = 0$.

Then the area (in sq. units) bounded by the curve y = (gof) (x) and the lines x = α, x = β and y = 0, is :

(JEE Main 2018)

(1) $\frac{1}{2}(\sqrt{2}-1)$ (2) $\frac{1}{2}(\sqrt{3}-1)$

(3) $\frac{1}{2}(\sqrt{3}+1)$ (4) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$

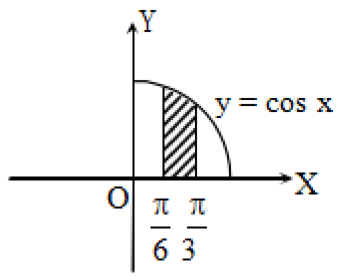
Sol: $18x^2 + 9\pi x + \pi^2 = 0$

⇒ (3x - π) (6x - π) = 0

$\Rightarrow \alpha = \frac{\pi}{6},\ \beta = \frac{\pi}{3}$

Also, gof (x) = cos x

⇒ The required area is $\int_{\pi/6}^{\pi/3} \cos x\,dx$



$$= \frac{\sqrt{3}-1}{2}$$

6. Area of the region bounded by the curve y = e^x and lines x = 0 and y = e is (JEE Advanced 2009)

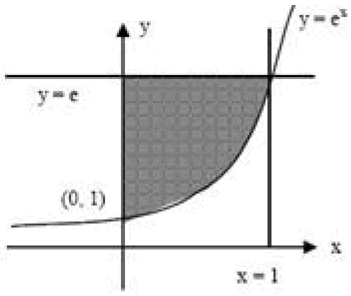
a. $\int_1^e \ln(e+1-y)\,dy$

b. e - 1

c. $e - \int_0^1 \ln e^x dx$

d. $\int_1^e \ln y\,dy$

Sol:



$$\text{Required Area} = \int_1^e \ln y\,dy$$

$$= (y \ln y - y)_1^e = (e - e) - \{-1\} = 1.$$

$$\text{Also, } \int_1^e \ln y\,dy = \int_1^e \ln\,(e+1-y)\,dy$$

$$\text{Further the required area} = e \times 1 - \int_0^1 e^x dx.$$

7. The area enclosed by the curves y = sin x + cos x and y = | cos x − sin x | over the interval [0, $\frac{\pi}{2}$] is (JEE Advanced 2013)

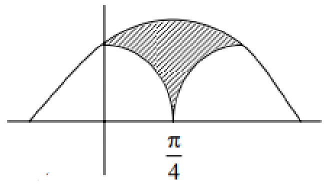
a. $4(\sqrt{2}-1)$

b. $2\sqrt{2}(\sqrt{2}-1)$

c. $2(\sqrt{2}-1)$

d. $2\sqrt{2}(\sqrt{2}+1)$

Sol:



$$\because\ y = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

$$y = \sqrt{2}\left|\sin\left(x-\frac{\pi}{4}\right)\right|$$

$$\text{Area of shaded part} = \int_0^{\pi/4}\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)-\sqrt{2}\left|\sin\left(x-\frac{\pi}{4}\right)\right|\right]dx$$

$$+\int_{\pi/4}^{\pi/2}\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)-\sqrt{2}\left|\sin\left(x-\frac{\pi}{4}\right)\right|\right]dx$$

$$= \sqrt{2}\left[\int_0^{\pi/4}\left(\sin\left(x+\frac{\pi}{4}\right)+\sin\left(x-\frac{\pi}{4}\right)\right)dx+\sqrt{2}\int_{\pi/4}^{\pi/2}\sin\left(x+\frac{\pi}{4}\right)-\sin\left(x-\frac{\pi}{4}\right)\right]$$

$$= 2-\sqrt{2}+2-\sqrt{2}$$

$$= 4-2\sqrt{2}$$

8. (JEE Advanced 2015)

Let $F(x)=\int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x\in\mathbb{R}$ and $f:\left[0,\frac{1}{2}\right]\to[0,\infty)$ be a continuous function.

For $a\in\left[0,\frac{1}{2}\right]$, if F′(a) + 2 is the area of the region bounded by x = 0, y = 0, y = f(x) and x = a, then f(0) is

Sol:

$$F(x) = \int_{x}^{x^2+\frac{\pi}{6}} (1+\cos 2t)\,dt$$

$$= t\Big|_{x}^{x^2+\frac{\pi}{6}} + \frac{\sin 2t}{2}\Big|_{x}^{x^2+\frac{\pi}{6}}$$

$$= x^2 - x + \frac{\pi}{6} + \frac{1}{2}\left[\sin\left(2x^2+\frac{\pi}{3}\right) - \sin 2x\right]$$

$$\therefore\ F'(x) = 2x - 1 + \frac{1}{2}\left[\cos\left(2x^2+\frac{\pi}{3}\right)\cdot 4x - 2\cos 2x\right]$$

From the question $\int_0^a f(x) = F'(a)+2$

Differentiating, we get

$f(a)=F''(a) \Rightarrow f(0)=F''(0)$

$$F''(0) = 4\cos^2\frac{\pi}{6} = 3 \qquad \therefore\ f(0) = 3$$

9. (JEE Advanced 2016)

Area of the region $\left\{(x, y) \in \mathbb{R}^2 : y \geq \sqrt{|x+3|}, 5y \leq x + 9 \leq 15\right\}$ is equal to

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

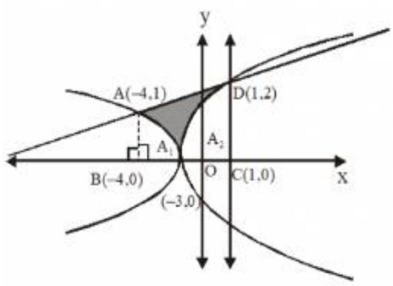
Clearly required area = area (trapezium ABCD) – $(A_1 + A_2)$ .....(i)

area (trapezium ABCD) $= \frac{1}{2}(1+2)(5) = \frac{15}{2}$

$$A_1 = \int_{-4}^{-3} \sqrt{-(x+3)}\,dx = \frac{2}{3}$$

and $A_2 = \int_{-3}^{1} (x+3)^{1/2}dx = \frac{16}{3}$

$\therefore$ From equation (1), we get required area $= \frac{15}{2} - \left(\frac{2}{3} + \frac{16}{3}\right) = \frac{3}{2}$



10. (JEE Advanced 2018)

Let $f : [0, \infty) \to R$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$ for all $x \in [0, \infty)$. Then, which of the following statement (s) is (are) TRUE?

(A) The curve $y = f(x)$ passes through the point (1, 2)

(B) The curve $y = f(x)$ passes through the point (2, –1)

(C) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi - 2}{4}$

(D) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi - 1}{4}$

$$e^{-x} f(x) = (1 - 2x)e^{-x} + \int_0^x e^{-t} f(t)\, dt$$

diff. $e^{-x} f'(x) - e^{-x} f(x) = -(1 - 2x) e^{-x} - 2e^{-x} + e^{-x} f(x)$

$f'(x) - 2 f(x) = -3 + 2x$

I.F. $= e^{-2x}$

Soln. $y \cdot e^{-2x} = C + \int e^{-2x} (2x - 3)\, dx$

$= C + 2 \int e^{-2x} \cdot x\, dx - 3 \cdot \int e^{-2x}\, dx$

$= C + 2\left[\frac{e^{-2x}}{-2} x - \frac{e^{-2x}}{4}\right] + \frac{3}{2} e^{-2x}$

$y = C \cdot e^{2x} - x - \frac{1}{2} + \frac{3}{2}$

$y = Ce^{2x} + 1 - x$

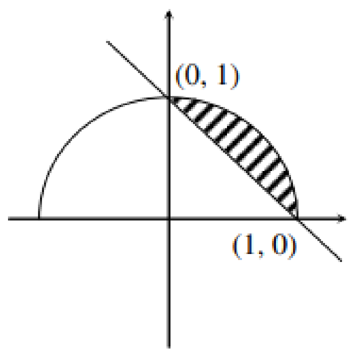
As $f(0) = 1$

$1 = C + 1$

$\Rightarrow C = 0$

$y = 1 - x \Rightarrow f(x) = 1 - x$

It passes (2, –1) (B)



$A = \frac{\pi(1)^2}{4} - \frac{1}{2} \times 1 \times 1$

$= \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$ (C)

11. (JEE Advanced 2018)

A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q (1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ $(n > 1)$. If the area of the region taken away by the farmer F_2 by the farmer F_2 is eaxtly 30% of the area of ΔPQR, then the value of n is

Sol:

$$\int_0^1 (x - x^n)\,dx = \frac{3}{10}$$

$$\left.\frac{x^2}{2}\right|_0^1 - \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{3}{10}$$

$$\frac{1}{2} - \frac{1}{n+1} = \frac{3}{10}$$

$$\frac{1}{n+1} = \frac{1}{2} - \frac{3}{10}$$

$$n + 1 = 5$$

$$n = 4$$