21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$ which are parallel to the line $x + 14y + 4 = 0$.

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles* if $8k^2 = 1$.

24. Find the equations of the tangent and normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0).

25. Find the equation of the tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$.

Choose the correct answer in Exercises 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at $x = 0$ is

(A) 3 (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{1}{3}$

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point

(A) (1, 2) (B) (2, 1) (C) (1, – 2) (D) (– 1, 2)

6.5 Approximations

In this section, we will use differentials to approximate values of certain quantities.

Let $f : D \to \mathbf{R}$, $D \subset \mathbf{R}$, be a given function and let $y = f(x)$. Let Δx denote a small increment in x. Recall that the increment in y corresponding to the increment in x, denoted by Δy, is given by $\Delta y = f(x + \Delta x) - f(x)$. We define the following

(i) The differential of x, denoted by dx, is defined by $dx = \Delta x$.

(ii) The differential of y, denoted by dy, is defined by $dy = f'(x)\ dx$ or
$$dy = \left(\frac{dy}{dx}\right)\Delta x.$$

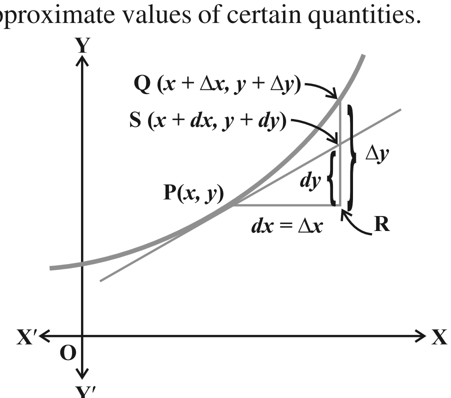

Fig 6.8

* Two curves intersect at right angle if the tangents to the curves at the point of intersection are perpendicular to each other.

In case $dx = \Delta x$ is relatively small when compared with x, dy is a good approximation of Δy and we denote it by $dy \approx \Delta y$.

For geometrical meaning of Δx, Δy, dx and dy, one may refer to Fig 6.8.

Note In view of the above discussion and Fig 6.8, we may note that the differential of the dependent variable is not equal to the increment of the variable where as the differential of independent variable is equal to the increment of the variable.

Example 21 Use differential to approximate $\sqrt{36.6}$.

Solution Take $y = \sqrt{x}$. Let $x = 36$ and let $\Delta x = 0.6$. Then

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x} = \sqrt{36.6} - \sqrt{36} = \sqrt{36.6} - 6$$

or $\sqrt{36.6} = 6 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{2\sqrt{x}}(0.6) = \frac{1}{2\sqrt{36}}(0.6) = 0.05 \qquad (\text{as } y = \sqrt{x})$$

Thus, the approximate value of $\sqrt{36.6}$ is $6 + 0.05 = 6.05$.

Example 22 Use differential to approximate $(25)^{\frac{1}{3}}$.

Solution Let $y = x^{\frac{1}{3}}$. Let $x = 27$ and let $\Delta x = -2$. Then

$$\Delta y = (x + \Delta x)^{\frac{1}{3}} - x^{\frac{1}{3}} = (25)^{\frac{1}{3}} - (27)^{\frac{1}{3}} = (25)^{\frac{1}{3}} - 3$$

or $(25)^{\frac{1}{3}} = 3 + \Delta y$

Now dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{3x^{\frac{2}{3}}}(-2) \qquad (\text{as } y = x^{\frac{1}{3}})$$

$$= \frac{1}{3((27)^{\frac{1}{3}})^2}(-2) = \frac{-2}{27} = -0.074$$

Thus, the approximate value of $(25)^{\frac{1}{3}}$ is given by

$$3 + (-0.074) = 2.926$$