Example 17 Find points on the curve $\frac{x^2}{4}+\frac{y^2}{25}=1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis.

Solution Differentiating $\frac{x^2}{4}+\frac{y^2}{25}=1$ with respect to x, we get

$$\frac{x}{2}+\frac{2y}{25}\frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = \frac{-25}{4}\frac{x}{y}$$

(i) Now, the tangent is parallel to the x-axis if the slope of the tangent is zero which gives $\frac{-25}{4}\frac{x}{y}=0$. This is possible if $x = 0$. Then $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $x = 0$ gives $y^2 = 25$, i.e., $y = \pm 5$.

Thus, the points at which the tangents are parallel to the x-axis are (0, 5) and (0, – 5).

(ii) The tangent line is parallel to y-axis if the slope of the normal is 0 which gives $\frac{4y}{25x}=0$, i.e., $y = 0$. Therefore, $\frac{x^2}{4}+\frac{y^2}{25}=1$ for $y = 0$ gives $x = \pm 2$. Hence, the points at which the tangents are parallel to the y-axis are (2, 0) and (–2, 0).

Example 18 Find the equation of the tangent to the curve $y=\frac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis.

Solution Note that on x-axis, $y = 0$. So the equation of the curve, when $y = 0$, gives $x = 7$. Thus, the curve cuts the x-axis at (7, 0). Now differentiating the equation of the curve with respect to x, we obtain

$$\frac{dy}{dx} = \frac{1-y(2x-5)}{(x-2)(x-3)} \qquad \text{(Why?)}$$

or
$$\left.\frac{dy}{dx}\right]_{(7,0)} = \frac{1-0}{(5)(4)}=\frac{1}{20}$$

Therefore, the slope of the tangent at (7, 0) is $\frac{1}{20}$. Hence, the equation of the tangent at (7, 0) is

$$y - 0 = \frac{1}{20}(x - 7) \quad \text{or} \quad 20y - x + 7 = 0$$

Example 19 Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ at (1, 1).

Solution Differentiating $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$ with respect to x, we get

$$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Therefore, the slope of the tangent at (1, 1) is $\left.\frac{dy}{dx}\right]_{(1,1)} = -1$.

So the equation of the tangent at (1, 1) is

$$y - 1 = -1\ (x - 1) \quad \text{or} \quad y + x - 2 = 0$$

Also, the slope of the normal at (1, 1) is given by

$$\frac{-1}{\text{slope of the tangent at (1,1)}} = 1$$

Therefore, the equation of the normal at (1, 1) is

$$y - 1 = 1\ (x - 1) \quad \text{or} \quad y - x = 0$$

Example 20 Find the equation of tangent to the curve given by

$$x = a\sin^3 t\,, \quad y = b\cos^3 t \qquad \text{... (1)}$$

at a point where $t = \frac{\pi}{2}$.

Solution Differentiating (1) with respect to t, we get

$$\frac{dx}{dt} = 3a\sin^2 t\cos t \quad \text{and} \quad \frac{dy}{dt} = -3b\cos^2 t\sin t$$

or $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t} = \frac{-b}{a}\frac{\cos t}{\sin t}$$

Therefore, slope of the tangent at $t = \frac{\pi}{2}$ is

$$\left.\frac{dy}{dx}\right]_{t=\frac{\pi}{2}} = \frac{-b\cos\frac{\pi}{2}}{a\sin\frac{\pi}{2}} = 0$$

Also, when $t = \frac{\pi}{2}$, $x = a$ and $y = 0$. Hence, the equation of tangent to the given curve at $t = \frac{\pi}{2}$, i.e., at $(a, 0)$ is

$$y - 0 = 0\,(x - a), \text{ i.e., } y = 0.$$

EXERCISE 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.
2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2},\ x \neq 2$ at $x = 10$.
3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose x-coordinate is 2.
4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x-coordinate is 3.
5. Find the slope of the normal to the curve $x = a\cos^3\theta,\ y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.
6. Find the slope of the normal to the curve $x = 1 - a\sin\theta, y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.
7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.
8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).