10. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.

12. Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing ?

(A) $(0,1)$ (B) $\left(\frac{\pi}{2}, \pi\right)$ (C) $\left(0, \frac{\pi}{2}\right)$ (D) None of these

14. Find the least value of a such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on $(1, 2)$.

15. Let I be any interval disjoint from $(-1, 1)$. Prove that the function f given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

16. Prove that the function f given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**.

19. The interval in which $y = x^2 e^{-x}$ is increasing is

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$

6.4 Tangents and Normals

In this section, we shall use differentiation to find the equation of the tangent line and the normal line to a curve at a given point.

Recall that the equation of a straight line passing through a given point (x_0, y_0) having finite slope m is given by

$$y - y_0 = m(x - x_0)$$

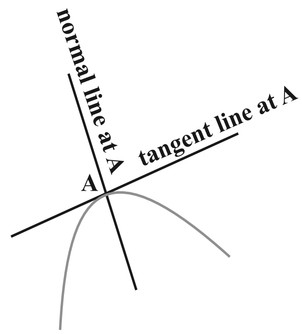

Fig 6.7

Note that the slope of the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by $\left.\dfrac{dy}{dx}\right]_{(x_0, y_0)} (= f'(x_0))$. So the equation of the tangent at (x_0, y_0) to the curve $y = f(x)$ is given by

$$y - y_0 = f'(x_0)(x - x_0)$$

Also, since the normal is perpendicular to the tangent, the slope of the normal to the curve $y = f(x)$ at (x_0, y_0) is $\dfrac{-1}{f'(x_0)}$, if $f'(x_0) \neq 0$. Therefore, the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is given by

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$$

i.e. $$(y - y_0) f'(x_0) + (x - x_0) = 0$$

Note If a tangent line to the curve $y = f(x)$ makes an angle θ with x-axis in the positive direction, then $\dfrac{dy}{dx}$ = slope of the tangent = $\tan\theta$.

Particular cases

(i) If slope of the tangent line is zero, then $\tan \theta = 0$ and so $\theta = 0$ which means the tangent line is parallel to the x-axis. In this case, the equation of the tangent at the point (x_0, y_0) is given by $y = y_0$.

(ii) If $\theta \to \dfrac{\pi}{2}$, then $\tan \theta \to \infty$, which means the tangent line is perpendicular to the x-axis, i.e., parallel to the y-axis. In this case, the equation of the tangent at (x_0, y_0) is given by $x = x_0$ (Why?).

Example 14 Find the slope of the tangent to the curve $y = x^3 - x$ at $x = 2$.

Solution The slope of the tangent at $x = 2$ is given by

$$\left.\frac{dy}{dx}\right]_{x=2} = 3x^2 - 1\Big]_{x=2} = 11.$$

Example 15 Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$.

Solution Slope of tangent to the given curve at (x, y) is

$$\frac{dy}{dx} = \frac{1}{2}(4x-3)^{\frac{-1}{2}} 4 = \frac{2}{\sqrt{4x-3}}$$

The slope is given to be $\frac{2}{3}$.

So $$\frac{2}{\sqrt{4x-3}} = \frac{2}{3}$$

or $$4x - 3 = 9$$

or $$x = 3$$

Now $y = \sqrt{4x-3} - 1$. So when $x = 3$, $y = \sqrt{4(3)-3} - 1 = 2$.

Therefore, the required point is (3, 2).

Example 16 Find the equation of all lines having slope 2 and being tangent to the curve $y + \frac{2}{x-3} = 0$.

Solution Slope of the tangent to the given curve at any point (x,y) is given by

$$\frac{dy}{dx} = \frac{2}{(x-3)^2}$$

But the slope is given to be 2. Therefore

$$\frac{2}{(x-3)^2} = 2$$

or $$(x-3)^2 = 1$$

or $$x - 3 = \pm 1$$

or $$x = 2, 4$$

Now $x = 2$ gives $y = 2$ and $x = 4$ gives $y = -2$. Thus, there are two tangents to the given curve with slope 2 and passing through the points (2, 2) and (4, – 2). The equation of tangent through (2, 2) is given by

$$y - 2 = 2(x - 2)$$

or $$y - 2x + 2 = 0$$

and the equation of the tangent through (4, – 2) is given by

$$y - (-2) = 2(x - 4)$$

or $$y - 2x + 10 = 0$$