Q50. Which of the following functions is decreasing in $\left(0, \frac{\pi}{2}\right)$?

(*a*) $\sin 2x$ (*b*) $\tan x$ (*c*) $\cos x$ (*d*) $\cos 3x$

Sol. Here, Let $f(x) = \cos x$; So, $f'(x) = -\sin x$

$f'(x) < 0$ in $\left(0, \frac{\pi}{2}\right)$

So $f(x) = \cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

Hence, the correct option is (*c*).

Q51. The function $f(x) = \tan x - x$

(*a*) always increases (*b*) always decreases

(*c*) never increases

(*d*) sometimes increases and sometimes decreases.

Sol. Here, $f(x) = \tan x - x$ So, $f'(x) = \sec^2 x - 1$

$f'(x) > 0 \ \forall \ x \in R$

So $f(x)$ is always increasing

Hence, the correct option is (*a*).

Q52. If x is real, the minimum value of $x^2 - 8x + 17$ is

(*a*) -1 (*b*) 0 (*c*) 1 (*d*) 2

Sol. Let $f(x) = x^2 - 8x + 17$

$f'(x) = 2x - 8$

For local maxima and local minima, $f'(x) = 0$

$\therefore$ $2x - 8 = 0 \Rightarrow x = 4$

So, $x = 4$ is the point of local maxima and local minima.

$f''(x) = 2 > 0$ minima at $x = 4$

$\therefore$ $f(x)_{x=4} = (4)^2 - 8(4) + 17$

$= 16 - 32 + 17 = 33 - 32 = 1$

So the minimum value of the function is 1

Hence, the correct option is (*c*).

Q53. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is:

(*a*) 126 (*b*) 0 (*c*) 135 (*d*) 160

Sol. Let $f(x) = x^3 - 18x^2 + 96x$; So, $f'(x) = 3x^2 - 36x + 96$

For local maxima and local minima $f'(x) = 0$

$\therefore$ $3x^2 - 36x + 96 = 0$

$\Rightarrow$ $x^2 - 12x + 32 = 0 \Rightarrow x^2 - 8x - 4x + 32 = 0$

$\Rightarrow$ $x(x - 8) - 4(x - 8) = 0 \Rightarrow (x - 8)(x - 4) = 0$

$\therefore$ $x = 8, 4 \in [0, 9]$

So, $x = 4, 8$ are the points of local maxima and local minima.

Now we will calculate the absolute maxima or absolute minima at $x = 0, 4, 8, 9$

$\therefore$ $f(x) = x^3 - 18x^2 + 96x$

$f(x)_{x=0} = 0 - 0 + 0 = 0$

$f(x)_{x=4} = (4)^3 - 18(4)^2 + 96(4)$
$= 64 - 288 + 384 = 448 - 288 = 160$
$f(x)_{x=8} = (8)^3 - 18(8)^2 + 96(8)$
$= 512 - 1152 + 768 = 1280 - 1152 = 128$
$f(x)_{x=9} = (9)^3 - 18(9)^2 + 96(9)$
$= 729 - 1458 + 864 = 1593 - 1458 = 135$

So, the absolute minimum value of f is 0 at $x = 0$

Hence, the correct option is (*b*).

Q54. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(*a*) two points of local maximum
(*b*) two points of local minimum
(*c*) one maxima and one minima
(*d*) no maxima or minima

Sol. We have $f(x) = 2x^3 - 3x^2 - 12x + 4$

$f'(x) = 6x^2 - 6x - 12$

For local maxima and local minima $f'(x) = 0$

$\therefore \quad 6x^2 - 6x - 12 = 0$

$\Rightarrow \quad x^2 - x - 2 = 0 \quad \Rightarrow \quad x^2 - 2x + x - 2 = 0$

$\Rightarrow \quad x(x-2) + 1(x-2) = 0 \quad \Rightarrow \quad (x+1)(x-2) = 0$

$\Rightarrow \quad x = -1, 2$ are the points of local maxima and local minima

Now $f''(x) = 12x - 6$

$f''(x)_{x=-1} = 12(-1) - 6 = -12 - 6 = -18 < 0$, maxima

$f''(x)_{x=2} = 12(2) - 6 = 24 - 6 = 18 > 0$ minima

So, the function is maximum at $x = -1$ and minimum at $x = 2$

Hence, the correct option is (*c*).

Q55. The maximum value of $\sin x \cos x$ is

(*a*) $\frac{1}{4}$ (*b*) $\frac{1}{2}$ (*c*) $\sqrt{2}$ (*d*) $2\sqrt{2}$

Sol. We have $f(x) = \sin x \cos x$

$\Rightarrow \quad f(x) = \frac{1}{2} \cdot 2 \sin x \cos x = \frac{1}{2} \sin 2x$

$f'(x) = \frac{1}{2} \cdot 2 \cos 2x$

$\Rightarrow \quad f'(x) = \cos 2x$

Now for local maxima and local minima $f'(x) = 0$

$\therefore \quad \cos 2x = 0$

$2x = (2n+1)\frac{\pi}{2}, \quad n \in I$

$\Rightarrow \quad x = (2n+1)\frac{\pi}{4}$

$$\therefore \quad x = \frac{\pi}{4}, \frac{3\pi}{4} \ldots$$

$$f''(x) = -2\sin 2x$$

$$f''(x)_{x=\frac{\pi}{4}} = -2\sin 2\cdot\frac{\pi}{4} = -2\sin\frac{\pi}{2} = -2 < 0 \text{ maxima}$$

$$f''(x)_{x=\frac{3\pi}{4}} = -2\sin 2\cdot\frac{3\pi}{4} = -2\sin\frac{3\pi}{2} = 2 > 0 \text{ minima}$$

So $f(x)$ is maximum at $x = \frac{\pi}{4}$

$\therefore$ Maximum value of $f(x) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2}$

Hence, the correct option is (*b*).

Q56. At $x = \frac{5\pi}{6}$, $f(x) = 2\sin 3x + 3\cos 3x$ is:

(*a*) maximum (*b*) minimum

(*c*) zero (*d*) neither maximum nor minimum.

Sol. We have $f(x) = 2\sin 3x + 3\cos 3x$

$$f'(x) = 2\cos 3x\cdot 3 - 3\sin 3x\cdot 3 = 6\cos 3x - 9\sin 3x$$

$$f''(x) = -6\sin 3x\cdot 3 - 9\cos 3x\cdot 3$$

$$= -18\sin 3x - 27\cos 3x$$

$$f''\left(\frac{5\pi}{6}\right) = -18\sin 3\left(\frac{5\pi}{6}\right) - 27\cos 3\left(\frac{5\pi}{6}\right)$$

$$= -18\sin\left(\frac{5\pi}{2}\right) - 27\cos\left(\frac{5\pi}{2}\right)$$

$$= -18\sin\left(2\pi + \frac{\pi}{2}\right) - 27\cos\left(2\pi + \frac{\pi}{2}\right)$$

$$= -18\sin\frac{\pi}{2} - 27\cos\frac{\pi}{2} = -18\cdot 1 - 27\cdot 0$$

$$= -18 < 0 \text{ maxima}$$

Maximum value of $f(x)$ at $x = \frac{5\pi}{6}$

$$f\left(\frac{5\pi}{6}\right) = 2\sin 3\left(\frac{5\pi}{6}\right) + 3\cos 3\left(\frac{5\pi}{6}\right) = 2\sin\frac{5\pi}{2} + 3\cos\frac{5\pi}{2}$$

$$= 2\sin\left(2\pi + \frac{\pi}{2}\right) + 3\cos\left(2\pi + \frac{\pi}{2}\right) = 2\sin\frac{\pi}{2} + 3\cos\frac{\pi}{2} = 2$$

Hence, the correct option is (a).

Q57. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

(*a*) 0 (*b*) 12 (*c*) 16 (*d*) 32

Sol. Given that $y = -x^3 + 3x^2 + 9x - 27$

$$\frac{dy}{dx} = -3x^2 + 6x + 9$$

$\therefore$ Slope of the given curve,

$$m = -3x^2 + 6x + 9 \qquad \left(\frac{dy}{dx} = m\right)$$

$$\frac{dm}{dx} = -6x + 6$$

For local maxima and local minima, $\frac{dm}{dx} = 0$

$\therefore$ $-6x + 6 = 0 \Rightarrow x = 1$

Now $\frac{d^2m}{dx^2} = -6 < 0$ maxima

$\therefore$ Maximum value of the slope at $x = 1$ is

$$m_{x=1} = -3(1)^2 + 6(1) + 9 = -3 + 6 + 9 = 12$$

Hence, the correct option is (*b*).

Q58. $f(x) = x^x$ has a stationary point at

(*a*) $x = e$ (*b*) $x = \frac{1}{e}$ (*c*) $x = 1$ (*d*) $x = \sqrt{e}$

Sol. We have $f(x) = x^x$

Taking log of both sides, we have

$$\log f(x) = x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = x \cdot \frac{1}{x} + \log x \cdot 1$$

$\Rightarrow$ $f'(x) = f(x)[1 + \log x] = x^x [1 + \log x]$

To find stationary point, $f'(x) = 0$

$\therefore$ $x^x[1 + \log x] = 0$

$x^x \neq 0$ $\therefore$ $1 + \log x = 0$

$\Rightarrow$ $\log x = -1 \Rightarrow x = e^{-1} \Rightarrow x = \frac{1}{e}$

Hence, the correct option is (*b*).

Q59. The maximum value of $\left(\frac{1}{x}\right)^x$ is:

(*a*) e (*b*) e^e (*c*) $e^{1/e}$ (*d*) $\left(\frac{1}{e}\right)^{1/e}$

Sol. Let $f(x) = \left(\frac{1}{x}\right)^x$

Taking log on both sides, we get

$$\log[f(x)] = x \log \frac{1}{x}$$

$\Rightarrow$ $\log[f(x)] = x \log x^{-1} \Rightarrow \log[f(x)] = -[x \log x]$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = -\left[x \cdot \frac{1}{x} + \log x \cdot 1\right] = -f(x)\,[1 + \log x]$$

$$\Rightarrow \qquad f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$$

For local maxima and local minima $f'(x) = 0$

$$-\left(\frac{1}{x}\right)^x [1 + \log x] = 0 \quad \Rightarrow \quad \left(\frac{1}{x}\right)^x [1 + \log x] = 0$$

$$\left(\frac{1}{x}\right)^x \neq 0$$

$$\therefore \qquad 1 + \log x = 0 \Rightarrow \log x = -1 \Rightarrow x = e^{-1}$$

So, $x = \frac{1}{e}$ is the stationary point.

Now $\quad f'(x) = -\left(\frac{1}{x}\right)^x [1 + \log x]$

$$f''(x) = -\left[\left(\frac{1}{x}\right)^x \left(\frac{1}{x}\right) + (1 + \log x) \cdot \frac{d}{dx}(x)^x\right]$$

$$f''(x) = -\left[(e)^{1/e}(e) + \left(1 + \log\frac{1}{e}\right)\frac{d}{\mathrm{dx}}\left(\frac{1}{e}\right)^{1/e}\right]$$

$$x = \frac{1}{e} = -e^{\frac{1}{e}} \;\; {}^{1} < 0 \text{ maxima}$$

$\therefore$ Maximum value of the function at $x = \frac{1}{e}$ is

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{1/e}\right)^{1/e} = e^{1/e}$$

Hence, the correct option is (c).

Fill in the blanks in each of the following exercises 60 to 64.

Q60. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point ____________ .

Sol. We have

$$y = 4x^2 + 2x - 8 \qquad \ldots(i)$$

and $$y = x^3 - x + 13 \qquad \ldots(ii)$$

Differentiating eq. (i) w.r.t. x, we have

$$\frac{dy}{dx} = 8x + 2 \quad \Rightarrow \quad m_1 = 8x + 2$$

[m is the slope of curve (i)]

Differentiating eq. (*ii*) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 1 \quad \Rightarrow \quad m_2 = 3x^2 - 1$$

[m_2 is the slope of curve (*ii*)]

If the two curves touch each other, then $m_1 = m_2$

$\therefore \quad 8x + 2 = 3x^2 - 1$

$\Rightarrow \quad 3x^2 - 8x - 3 = 0 \Rightarrow 3x^2 - 9x + x - 3 = 0$

$\Rightarrow \quad 3x(x - 3) + 1(x - 3) = 0 \Rightarrow (x - 3)(3x + 1) = 0$

$\therefore \quad x = 3, \frac{-1}{3}$

Putting $x = 3$ in eq. (*i*), we get

$$y = 4(3)^2 + 2(3) - 8 = 36 + 6 - 8 = 34$$

So, the required point is (3, 34)

Now for $x = -\frac{1}{3}$

$$y = 4\left(\frac{-1}{3}\right)^2 + 2\left(\frac{-1}{3}\right) - 8 = 4 \times \frac{1}{9} - \frac{2}{3} - 8$$

$$= \frac{4}{9} - \frac{2}{3} - 8 = \frac{4 - 6 - 72}{9} = \frac{-74}{9}$$

$\therefore$ Other required point is $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

Hence, the required points are (3, 34) and $\left(-\frac{1}{3}, \frac{-74}{9}\right)$.

Q61. The equation of normal to the curve $y = \tan x$ at (0, 0) is ____________ .

Sol. We have $y = \tan x$. So, $\frac{dy}{dx} = \sec^2 x$

$\therefore$ Slope of the normal $= \frac{-1}{\sec^2 x} = -\cos^2 x$

at the point (0, 0) the slope $= -\cos^2(0) = -1$

So the equation of normal at (0, 0) is $y - 0 = -1(x - 0)$

$\Rightarrow \quad y = -x \Rightarrow y + x = 0$

Hence, the required equation is $y + x = 0$.

Q62. The values of a for which the function $f(x) = \sin x - ax + b$ increases on **R** are ____________ .

Sol. We have $f(x) = \sin x - ax + b \Rightarrow f'(x) = \cos x - a$

For increasing the function $f'(x) > 0$

$\therefore \quad \cos x - a > 0$

Since $\cos x \in [-1, 1]$

$\therefore \qquad a < -1 \quad \Rightarrow \quad a \in (-\infty, -1)$

Hence, the value of a is $(-\infty, -1)$.

Q63. The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, $x > 0$, decreases in the interval ____________ .

Sol. We have $f(x) = \dfrac{2x^2 - 1}{x^4}$

$$f'(x) = \frac{x^4(4x) - (2x^2 - 1)\cdot 4x^3}{x^8}$$

$$\Rightarrow f'(x) = \frac{4x^5 - (2x^2 - 1)\cdot 4x^3}{x^8} = \frac{4x^3[x^2 - 2x^2 + 1]}{x^8} = \frac{4(-x^2 + 1)}{x^5}$$

For decreasing the function $f'(x) < 0$

$$\therefore \qquad \frac{4(-x^2 + 1)}{x^5} < 0 \quad \Rightarrow \quad -x^2 + 1 < 0 \quad \Rightarrow \quad x^2 > 1$$

$$\therefore \qquad x > \pm 1 \quad \Rightarrow \quad x \in (1, \infty)$$

Hence, the required interval is $(1, \infty)$.

Q64. The least value of the function $f(x) = ax + \dfrac{b}{x}$ (where $a > 0$, $b > 0, x > 0$) is ____________ .

Sol. Here, $\qquad f(x) = ax + \dfrac{b}{x} \quad \Rightarrow f'(x) = a - \dfrac{b}{x^2}$

For maximum and minimum value $f'(x) = 0$

$$\therefore \qquad a - \frac{b}{x^2} = 0 \quad \Rightarrow \quad x^2 = \frac{b}{a} \quad \Rightarrow \quad x = \pm\sqrt{\frac{b}{a}}$$

Now $\qquad f''(x) = \dfrac{2b}{x^3}$

$$f''(x)_{x=\sqrt{\frac{b}{a}}} = \frac{2b}{\left(\frac{b}{a}\right)^{3/2}} = 2\frac{a^{3/2}}{b^{1/2}} > 0 \qquad (\because \quad a, b > 0)$$

Hence, minima

So the least value of the function at $x = \sqrt{\dfrac{b}{a}}$ is

$$f\left(\sqrt{\frac{b}{a}}\right) = a\cdot\sqrt{\frac{b}{a}} + \frac{b}{\sqrt{\frac{b}{a}}} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

Hence, least value is $2\sqrt{ab}$.