Q22. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Sol. Given that: $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left(0, \frac{\pi}{4}\right)$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{1}{1+(\sin x+\cos x)^2}\cdot\frac{d}{dx}(\sin x+\cos x)$$

$$\Rightarrow f'(x) = \frac{1\times(\cos x-\sin x)}{1+(\sin x+\cos x)^2}$$

$$\Rightarrow f'(x) = \frac{\cos x-\sin x}{1+\sin^2 x+\cos^2 x+2\sin x\cos x}$$

$$\Rightarrow f'(x) = \frac{\cos x-\sin x}{1+1+2\sin x\cos x} \quad \Rightarrow f'(x) = \frac{\cos x-\sin x}{2+2\sin x\cos x}$$

For an increasing function $f'(x) \geq 0$

$$\therefore \quad \frac{\cos x-\sin x}{2+2\sin x\cos x} \geq 0$$

$$\Rightarrow \quad \cos x - \sin x \geq 0 \quad \left[\because \ (2+\sin 2x) \geq 0 \text{ in } \left(0, \frac{\pi}{4}\right)\right]$$

$\Rightarrow$ $\cos x \geq \sin x$, which is true for $\left(0, \frac{\pi}{4}\right)$

Hence, the given function $f(x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

Q23. At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum ? Also find the maximum slope.

Sol. Given that: $y = -x^3 + 3x^2 + 9x - 27$

Differentiating both sides w.r.t. x, we get $\frac{dy}{dx} = -3x^2 + 6x + 9$

Let slope of the cuve $\frac{dy}{dx} = Z$

$$\therefore \quad z = -3x^2 + 6x + 9$$

Differentiating both sides w.r.t. x, we get $\frac{dz}{dx} = -6x + 6$

For local maxima and local minima, $\frac{dz}{dx} = 0$

$$\therefore \quad -6x + 6 = 0 \quad \Rightarrow \quad x = 1$$

$$\Rightarrow \quad \frac{d^2z}{dx^2} = -6 < 0 \quad \text{Maxima}$$

Put $x = 1$ in equation of the curve $y = (-1)^3 + 3(1)^2 + 9(1) - 27$

$$= -1 + 3 + 9 - 27 = -16$$

Maximum slope = $-3(1)^2 + 6(1) + 9 = 12$

Hence, $(1, -16)$ is the point at which the slope of the given curve is maximum and maximum slope = 12.

Q24. Prove that $f(x) = \sin x + \sqrt{3}\cos x$ has maximum value at $x = \frac{\pi}{6}$.

Sol. We have: $f(x) = \sin x + \sqrt{3}\cos x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

$$= 2\left(\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x\right) = 2\sin\left(x + \frac{\pi}{3}\right)$$

$$f'(x) = 2\cos\left(x + \frac{\pi}{3}\right);\ f''(x) = -2\sin\left(x + \frac{\pi}{3}\right)$$

$$f''(x)_{x=\frac{\pi}{6}} = -2\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$$

$$= -2\sin\frac{\pi}{2} = -2.1 = -2 < 0 \text{ (Maxima)}$$

$$= -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \text{ (Maxima)}$$

Maximum value of the function at $x = \frac{\pi}{6}$ is

$$\sin\frac{\pi}{6} + \sqrt{3}\cos\frac{\pi}{6} = \frac{1}{2} + \sqrt{3}\,.\,\frac{\sqrt{3}}{2} = 2$$

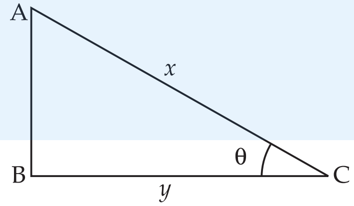
Hence, the given function has maximum value at $x = \frac{\pi}{6}$ and the maximum value is 2.

LONG ANSWER TYPE QUESTIONS

Q25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

Sol. Let ΔABC be the right angled triangle in which $\angle B = 90°$

Let $AC = x$, $BC = y$



$\therefore \quad AB = \sqrt{x^2 - y^2}$

$\angle ACB = \theta$

Let $\quad Z = x + y \quad$ (given)

Now area of ΔABC, $A = \frac{1}{2} \times AB \times BC$

$\Rightarrow \quad A = \frac{1}{2} y \cdot \sqrt{x^2 - y^2} \quad \Rightarrow A = \frac{1}{2} y \cdot \sqrt{(Z - y)^2 - y^2}$

Squaring both sides, we get

$$A^2 = \frac{1}{4} y^2 \left[(Z - y)^2 - y^2\right] \Rightarrow A^2 = \frac{1}{4} y^2 [Z^2 + y^2 - 2Zy - y^2]$$

$$\Rightarrow \quad P = \frac{1}{4} y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4} [y^2 Z^2 - 2Zy^3] \qquad [A^2 = P]$$

Differentiating both sides w.r.t. y we get

$$\frac{dP}{dy} = \frac{1}{4} [2yZ^2 - 6Zy^2] \qquad ...(i)$$

For local maxima and local minima, $\frac{dP}{dy} = 0$

$\therefore \quad \frac{1}{4} (2yZ^2 - 6Zy^2) = 0$

$\Rightarrow \quad \frac{2yZ}{4} (Z - 3y) = 0 \quad \Rightarrow \quad yZ(Z - 3y) = 0$

$\Rightarrow \quad yZ \neq 0 \qquad (\because \quad y \neq 0 \text{ and } Z \neq 0)$

$\therefore \quad Z - 3y = 0$

$\Rightarrow \quad y = \frac{Z}{3} \quad \Rightarrow y = \frac{x + y}{3} \qquad (\because \quad Z = x + y)$

$\Rightarrow \quad 3y = x + y \quad \Rightarrow 3y - y = x \quad \Rightarrow 2y = x$

$\Rightarrow \quad \frac{y}{x} = \frac{1}{2} \quad \Rightarrow \cos\theta = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}$

Differentiating eq. (i) w.r.t. y, we have $\frac{d^2P}{dy^2} = \frac{1}{4} [2Z^2 - 12Zy]$

$$\frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} = \frac{1}{4}\left[2Z^2 - 12Z \cdot \frac{Z}{3}\right]$$

$$= \frac{1}{4}[2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima}$$

Hence, the area of the given triangle is maximum when the angle between its hypotenuse and a side is $\frac{\pi}{3}$.

Q26. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.

Sol. We have $\quad f(x) = x^5 - 5x^4 + 5x^3 - 1$

$\Rightarrow \quad f'(x) = 5x^4 - 20x^3 + 15x^2$

For local maxima and local minima, $f'(x) = 0$

$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0 \quad \Rightarrow 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0 \quad \Rightarrow x^2(x - 3)(x - 1) = 0$

$\therefore \quad x = 0, x = 1$ and $x = 3$

Now $\quad f''(x) = 20x^3 - 60x^2 + 30x$

$\Rightarrow \quad f''(x)_{\text{at } x=0} = 20(0)^3 - 60(0)^2 + 30(0) = 0$ which is neither maxima nor minima.

$\therefore \quad f(x)$ has the point of inflection at $x = 0$

$$f''(x)_{\text{at } x=1} = 20(1)^3 - 60(1)^2 + 30(1)$$
$$= 20 - 60 + 30 = -10 < 0 \text{ Maxima}$$
$$f''(x)_{\text{at } x=3} = 20(3)^3 - 60(3)^2 + 30(3)$$
$$= 540 - 540 + 90 = 90 > 0 \text{ Minima}$$

The maximum value of the function at $x = 1$

$$f(x) = (1)^5 - 5(1)^4 + 5(1)^3 - 1$$
$$= 1 - 5 + 5 - 1 = 0$$

The minimum value at $x = 3$ is

$$f(x) = (3)^5 - 5(3)^4 + 5(3)^3 - 1$$
$$= 243 - 405 + 135 - 1 = 378 - 406 = -28$$

Hence, the function has its maxima at $x = 1$ and the maximum value = 0 and it has minimum value at x = 3 and its minimum value is – 28.

$x = 0$ is the point of inflection.

Q27. A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300 per subscriber per year. The company proposes to increase the annual subscription and it is believed that for every increase of ₹ 1.00, one subscriber will discontinue the service. Find what increase will bring maximum profit?

Sol. Let us consider that the company increases the annual subscription by ₹ x.

So, x is the number of subscribers who discontinue the services.

$\therefore \quad$ Total revenue, $\quad R(x) = (500 - x)(300 + x)$

$$= 150000 + 500x - 300x - x^2$$
$$= -x^2 + 200x + 150000$$

Differentiating both sides w.r.t. x, we get $\quad R'(x) = -2x + 200$

For local maxima and local minima, $R'(x) = 0$

$$-2x + 200 = 0 \quad \Rightarrow \quad x = 100$$
$$R''(x) = -2 < 0 \text{ Maxima}$$

So, $R(x)$ is maximum at $x = 100$

Hence, in order to get maximum profit, the company should increase its annual subscription by ₹ 100.

Q28. If the straight line $x \cos \alpha + y \sin \alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$.

Sol. The given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(*i*)

and the straight line $x \cos \alpha + y \sin \alpha = p$ ...(*ii*)

Differentiating eq. (*i*) w.r.t. x, we get

$$\frac{1}{a^2} \cdot 2x + \frac{1}{b^2} \cdot 2y \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{x}{y}$$

So the slope of the curve $= \frac{-b^2}{a^2} \cdot \frac{x}{y}$

Now differentiating eq. (*ii*) w.r.t. x, we have

$$\cos \alpha + \sin \alpha \cdot \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \frac{-\cos \alpha}{\sin \alpha} = -\cot \alpha$$

So, the slope of the straight line = – cot α

If the line is the tangent to the curve, then

$$\frac{-b^2}{a^2} \cdot \frac{x}{y} = -\cot \alpha \quad \Rightarrow \quad \frac{x}{y} = \frac{a^2}{b^2} \cdot \cot \alpha \quad \Rightarrow \quad x = \frac{a^2}{b^2} \cot \alpha \cdot y$$

Now from eq. (*ii*) we have $x \cos \alpha + y \sin \alpha = p$

$$\Rightarrow \quad \frac{a^2}{b^2} \cdot \cot \alpha \cdot y \cdot \cos \alpha + y \sin \alpha = p$$

$$\Rightarrow \quad a^2 \cot \alpha \cdot \cos \alpha \, y + b^2 \sin \alpha \, y = b^2 p$$

$$\Rightarrow \quad a^2 \frac{\cos \alpha}{\sin \alpha} \cdot \cos \alpha \, y + b^2 \sin \alpha \, y = b^2 p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha \, y + b^2 \sin^2 \alpha \, y = b^2 \sin \alpha \, p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = \frac{b^2}{y} \cdot \sin \alpha \cdot p$$

$$\Rightarrow \quad a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p \cdot p \quad \left[\because \frac{b^2}{y} \sin \alpha = p\right]$$

Hence, $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$

Alternate method:

We know that $y = mx + c$ will touch the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if } c^2 = a^2m^2 + b^2$$

Here equation of straight line is $x \cos\alpha + y \sin\alpha = p$ and that of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x \cos\alpha + y \sin\alpha = p$$

$$\Rightarrow \quad y \sin\alpha = -x \cos\alpha + p$$

$$\Rightarrow \quad y = -x \frac{\cos\alpha}{\sin\alpha} + \frac{p}{\sin\alpha} \quad \Rightarrow y = -x \cot\alpha + \frac{p}{\sin\alpha}$$

Comparing with $y = mx + c$, we get

$$m = -\cot\alpha \quad \text{and} \quad c = \frac{p}{\sin\alpha}$$

So, according to the condition, we get $c^2 = a^2m^2 + b^2$

$$\frac{p^2}{\sin^2\alpha} = a^2(-\cot\alpha)^2 + b^2$$

$$\Rightarrow \quad \frac{p^2}{\sin^2\alpha} = \frac{a^2\cos^2\alpha}{\sin^2\alpha} + b^2 \quad \Rightarrow p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$$

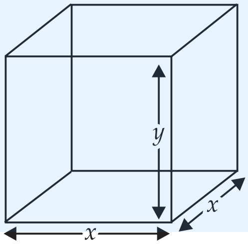
Hence, $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$ Hence proved.

Q29. An open box with square base is to be made of a given quantity of card board of area c^2. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Sol. Let x be the length of the side of the square base of the cubical open box and y be its height.

$\therefore$ Surface area of the open box

$$c^2 = x^2 + 4xy \quad \Rightarrow y = \frac{c^2 - x^2}{4x} \quad \ldots(i)$$



Now volume of the box, $V = x \times x \times y$

$$\Rightarrow \quad V = x^2y$$

$$\Rightarrow \quad V = x^2\left(\frac{c^2 - x^2}{4x}\right)$$

$$\Rightarrow \quad V = \frac{1}{4}(c^2x - x^3)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \frac{1}{4}(c^2 - 3x^2) \qquad ...(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$$\therefore \quad \frac{1}{4}(c^2 - 3x^2) = 0 \Rightarrow c^2 - 3x^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{c^2}{3}$$

$$\therefore \quad x = \sqrt{\frac{c^2}{3}} = \frac{c}{\sqrt{3}}$$

Now again differentiating eq. (*ii*) w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{1}{4}(-6x) = \frac{-3}{2} \cdot \frac{c}{\sqrt{3}} < 0 \quad \text{(maxima)}$$

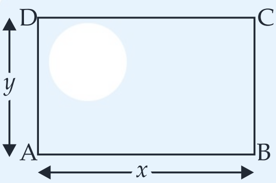
Volume of the cubical box (V) = $x^2 y$

$$= x^2\left(\frac{c^2 - x^2}{4x}\right) = \frac{c}{\sqrt{3}}\left[\frac{c^2 - \frac{c^2}{3}}{4}\right] = \frac{c}{\sqrt{3}} \times \frac{2c^2}{3 \times 4} = \frac{c^3}{6\sqrt{3}}$$

Hence, the maximum volume of the open box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Q30. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its sides. Also find the maximum volume.

Sol. Let x and y be the length and breadth of a given rectangle ABCD as per question, the rectangle be revolved about side AD which will make a cylinder with radius x and height y.



$\therefore$ Volume of the cylinder $V = \pi r^2 h$

$$\Rightarrow \quad V = \pi x^2 y \qquad ...(i)$$

Now perimeter of rectangle $P = 2(x + y) \Rightarrow 36 = 2(x + y)$

$$\Rightarrow \quad x + y = 18 \Rightarrow y = 18 - x \qquad ...(ii)$$

Putting the value of y in eq. (*i*) we get

$$V = \pi x^2(18 - x)$$

$$\Rightarrow \quad V = \pi(18x^2 - x^3)$$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = \pi(36x - 3x^2) \qquad ...(iii)$$

For local maxima and local minima $\frac{dV}{dx} = 0$

$\therefore \quad \pi(36x - 3x^2) = 0 \quad \Rightarrow 36x - 3x^2 = 0$

$\Rightarrow \quad 3x(12 - x) = 0$

$\Rightarrow \quad x \neq 0 \quad \therefore \quad 12 - x = 0 \Rightarrow x = 12$

From eq. (*ii*) $y = 18 - 12 = 6$

Differentiating eq. (*iii*) w.r.t. x, we get $\frac{d^2V}{dx^2} = \pi(36 - 6x)$

at $x = 12$ $\quad \frac{d^2V}{dx^2} = \pi(36 - 6 \times 12)$

$= \pi(36 - 72) = -36\pi < 0$ maxima

Now volume of the cylinder so formed $= \pi x^2 y$

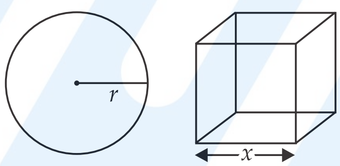
$= \pi \times (12)^2 \times 6 = \pi \times 144 \times 6 = 864\pi \text{ cm}^3$

Hence, the required dimensions are 12 cm and 6 cm and the maximum volume is $864\pi \text{ cm}^3$.

Q31. If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum?

Sol. Let x be the edge of the cube and r be the radius of the sphere.

Surface area of cube $= 6x^2$



and surface area of the sphere $= 4\pi r^2$

$\therefore \quad 6x^2 + 4\pi r^2 = K(\text{constant}) \quad \Rightarrow \quad r = \sqrt{\frac{K - 6x^2}{4\pi}} \quad ...(i)$

Volume of the cube $= x^3$ and the volume of sphere $= \frac{4}{3}\pi r^3$

$\therefore$ Sum of their volumes (V) = Volume of cube + Volume of sphere

$\Rightarrow \quad V = x^3 + \frac{4}{3}\pi r^3$

$\Rightarrow \quad V = x^3 + \frac{4}{3}\pi \times \left(\frac{K - 6x^2}{4\pi}\right)^{3/2}$

Differentiating both sides w.r.t. x, we get

$$\frac{dV}{dx} = 3x^2 + \frac{4\pi}{3} \times \frac{3}{2}(K - 6x^2)^{1/2}(-12x) \times \frac{1}{(4\pi)^{3/2}}$$

$$= 3x^2 + \frac{2\pi}{(4\pi)^{3/2}} \times (-12x)\,(K - 6x^2)^{1/2}$$

$$= 3x^2 + \frac{1}{4\pi^{1/2}} \times (-12x)\,(K - 6x^2)^{1/2}$$

$$\therefore \quad \frac{dV}{dx} = 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} \qquad ...(ii)$$

For local maxima and local minima, $\frac{dV}{dx} = 0$

$$\therefore \quad 3x^2 - \frac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2} = 0$$

$$\Rightarrow \quad 3x\left[x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}\right] = 0$$

$$x \neq 0 \quad \therefore \quad x - \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}} = 0$$

$$\Rightarrow \quad x = \frac{(K - 6x^2)^{1/2}}{\sqrt{\pi}}$$

Squaring both sides, we get

$$x^2 = \frac{K - 6x^2}{\pi} \quad \Rightarrow \pi x^2 = K - 6x^2$$

$$\Rightarrow \quad \pi x^2 + 6x^2 = K \quad \Rightarrow x^2(\pi + 6) = K \quad \Rightarrow \quad x^2 = \frac{K}{\pi + 6}$$

$$\therefore \quad x = \sqrt{\frac{K}{\pi + 6}}$$

Now putting the value of K in eq. (i), we get

$$6x^2 + 4\pi r^2 = x^2(\pi + 6)$$

$$\Rightarrow \quad 6x^2 + 4\pi r^2 = \pi x^2 + 6x^2 \quad \Rightarrow 4\pi r^2 = \pi x^2 \quad \Rightarrow \quad 4r^2 = x^2$$

$$\therefore \quad 2r = x$$

$$\therefore \quad x : 2r = 1 : 1$$

Now differentiating eq. (ii) w.r.t x, we have

$$\frac{d^2V}{dx^2} = 6x - \frac{3}{\sqrt{\pi}}\frac{d}{dx}[x(K - 6x^2)^{1/2}]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1}{2\sqrt{K - 6x^2}} \times (-12x) + (K - 6x^2)^{1/2} \cdot 1\right]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2}{\sqrt{K - 6x^2}} + \sqrt{K - 6x^2}\right]$$

$$= 6x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + K - 6x^2}{\sqrt{K - 6x^2}}\right] = 6x + \frac{3}{\sqrt{\pi}}\left[\frac{12x^2 - K}{\sqrt{K - 6x^2}}\right]$$

Put $x = \sqrt{\frac{K}{\pi + 6}} = 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}}\left[\frac{\frac{12K}{\pi+6} - K}{\sqrt{K - \frac{6K}{\pi+6}}}\right]$

$$= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}}\left[\frac{12K - \pi K - 6K}{\sqrt{\frac{\pi K + 6K - 6K}{\pi + 6}}}\right]$$

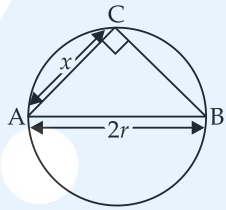
$$= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\sqrt{\pi}}\left[\frac{6K - \pi K}{\sqrt{\frac{\pi K}{\pi+6}}}\right]$$

$$= 6\sqrt{\frac{K}{\pi+6}} + \frac{3}{\pi\sqrt{K}}[(6K - \pi K)\sqrt{\pi + 6}] > 0$$

So it is minima.

Hence, the required ratio is 1 : 1 when the combined volume is minimum.

Q32. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles.



Sol. Let AB be the diameter and C be any point on the circle with radius r.

$\angle ACB = 90°$ [angle in the semi circle is 90°]

Let $AC = x$

$\therefore \quad BC = \sqrt{AB^2 - AC^2}$

$\Rightarrow \quad BC = \sqrt{(2r)^2 - x^2} \quad \Rightarrow BC = \sqrt{4r^2 - x^2}$ ...(*i*)

Now area of ΔABC, $A = \frac{1}{2} \times AC \times BC$

$\Rightarrow \quad A = \frac{1}{2}x \cdot \sqrt{4r^2 - x^2}$

Squaring both sides, we get

$$A^2 = \frac{1}{4}x^2(4r^2 - x^2)$$

Let $A^2 = Z$

$\therefore \quad Z = \frac{1}{4}x^2(4r^2 - x^2) \quad \Rightarrow Z = \frac{1}{4}(4x^2r^2 - x^4)$

Differentiating both sides w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{1}{4}[8xr^2 - 4x^3] \quad ...(ii)$$

For local maxima and local minima $\frac{dZ}{dx} = 0$

$\therefore \quad \frac{1}{4}[8xr^2 - 4x^3] = 0 \quad \Rightarrow \quad x[2r^2 - x^2] = 0$

$x \neq 0 \quad \therefore \quad 2r^2 - x^2 = 0$

$\Rightarrow \quad x^2 = 2r^2 \quad \Rightarrow \quad x = \sqrt{2}r = AC$

Now from eq. (i) we have

$$BC = \sqrt{4r^2 - 2r^2} \quad \Rightarrow BC = \sqrt{2r^2} \quad \Rightarrow BC = \sqrt{2}r$$

So $\quad AC = BC$

Hence, ΔABC is an isosceles triangle.

Differentiating eq. (ii) w.r.t. x, we get $\frac{d^2Z}{dx^2} = \frac{1}{4}[8r^2 - 12x^2]$

Put $x = \sqrt{2}r$

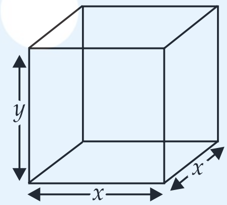
$$\therefore \quad \frac{d^2Z}{dx^2} = \frac{1}{4}[8r^2 - 12 \times 2r^2] = \frac{1}{4}[8r^2 - 24r^2]$$

$$= \frac{1}{4} \times (-16r^2) = -4r^2 < 0 \quad \text{maxima}$$

Hence, the area of ΔABC is maximum when it is an isosceles triangle.

Q33. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and botttom costs ₹ 5/cm^2 and the material for the sides costs ₹ 2.50/cm^2. Find the least cost of the box.

Sol. Let x be the side of the square base and y be the length of the vertical sides.



Area of the base and bottom = $2x^2$ cm^2

$\therefore$ Cost of the material required = ₹ $5 \times 2x^2$

= ₹ $10x^2$

Area of the 4 sides = $4xy$ cm^2

$\therefore$ Cost of the material for the four sides

= ₹ $2.50 \times 4xy$ = ₹ $10xy$

Total cost $\quad C = 10x^2 + 10xy \quad ...(i)$

New volume of the box = $x \times x \times y$

$\Rightarrow \quad 1024 = x^2y$

$$\therefore \quad y = \frac{1024}{x^2} \quad ...(ii)$$

Putting the value of y in eq. (i) we get

$$C = 10x^2 + 10x \times \frac{1024}{x^2} \quad \Rightarrow C = 10x^2 + \frac{10240}{x}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 20x - \frac{10240}{x^2} \quad ...(iii)$$

For local maxima and local minima $\frac{dC}{dx} = 0$

$$20 - \frac{10240}{x^2} = 0$$

$\Rightarrow \quad 20x^3 - 10240 = 0 \quad \Rightarrow x^3 = 512 \quad \Rightarrow \quad x = 8$ cm

Now from eq. (ii)

$$y = \frac{10240}{(8)^2} = \frac{10240}{64} = 16 \text{ cm}$$

$\therefore$ Cost of material used $C = 10x^2 + 10xy$

$= 10 \times 8 \times 8 + 10 \times 8 \times 16 = 640 + 1280 = 1920$

Now differentiating eq. (iii) we get

$$\frac{d^2C}{dx^2} = 20 + \frac{20480}{x^3}$$

Put $x = 8$

$$= 20 + \frac{20480}{(8)^3} = 20 + \frac{20480}{512} = 20 + 40 = 60 > 0 \text{ minima}$$

Hence, the required cost is ₹ 1920 which is the minimum.

Q34. The sum of the surface areas of a rectangular parallelopiped with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of the sphere. Also find the minimum value of the sum of their volumes.

Sol. Let 'r' be the radius of the sphere.

$\therefore$ Surface area of the sphere $= 4\pi r^2$

Volume of the sphere $= \frac{4}{3}\pi r^3$

The sides of the parallelopiped are x, $2x$ and $\frac{x}{3}$

$$\therefore \quad \text{Its surface area} = 2\left[x \times 2x + 2x \times \frac{x}{3} + x \times \frac{x}{3}\right]$$

$$= 2\left[2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right] = 2[2x^2 + x^2]$$

$$= 2[3x^2] = 6x^2$$

Volume of the parallelopiped = $x \times 2x \times \dfrac{x}{3} = \dfrac{2}{3}x^3$

As per the conditions of the question,

Surface area of the parallelopiped

+ Surface area of the sphere = constant

$\Rightarrow \quad 6x^2 + 4\pi r^2 = \text{K (constant)} \quad \Rightarrow 4\pi r^2 = \text{K} - 6x^2$

$$\therefore \quad r^2 = \frac{\text{K} - 6x^2}{4\pi} \quad ...(i)$$

Now let V = Volume of parallelopiped + Volume of the sphere

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi\left[\frac{\text{K} - 6x^2}{4\pi}\right]^{3/2} \quad \text{[from eq. } (i)]$$

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{(4)^{3/2}\pi^{3/2}}[\text{K} - 6x^2]^{3/2}$$

$$\Rightarrow \quad \text{V} = \frac{2}{3}x^3 + \frac{4}{3}\pi \times \frac{1}{8 \times \pi^{3/2}}[\text{K} - 6x^2]^{3/2}$$

$$\Rightarrow \quad = \frac{2}{3}x^3 + \frac{1}{6\sqrt{\pi}}[\text{K} - 6x^2]^{3/2}$$

Differentiating both sides w.r.t. x, we have

$$\frac{d\text{V}}{dx} = \frac{2}{3}\cdot 3x^2 + \frac{1}{6\sqrt{\pi}}\left[\frac{3}{2}(K - 6x^2)^{1/2}(-12x)\right]$$

$$= 2x^2 + \frac{1}{6\sqrt{\pi}} \times \frac{3}{2} \times (-12x)(\text{K} - 6x^2)^{1/2}$$

$$= 2x^2 - \frac{3x}{\sqrt{\pi}}[\text{K} - 6x^2)^{1/2}$$

For local maxima and local minima, we have $\dfrac{d\text{V}}{dx} = 0$

$$\therefore \quad 2x^2 - \frac{3x}{\sqrt{\pi}}(\text{K} - 6x^2)^{1/2} = 0$$

$$\Rightarrow \quad 2\sqrt{\pi}x^2 - 3x(\text{K} - 6x^2)^{1/2} = 0$$

$$\Rightarrow \quad x[2\sqrt{\pi}x - 3(\text{K} - 6x^2)^{1/2}] = 0$$

Here $x \neq 0$ and $2\sqrt{\pi}x - 3(\text{K} - 6x^2)^{1/2} = 0$

$$\Rightarrow \quad 2\sqrt{\pi}x = 3(\text{K} - 6x^2)^{1/2}$$

Squaring both sides, we get

$$4\pi x^2 = 9(\text{K} - 6x^2) \Rightarrow 4\pi x^2 = 9\text{K} - 54x^2$$

$$\Rightarrow \quad 4\pi x^2 + 54x^2 = 9K$$

$$\Rightarrow \quad K = \frac{4\pi x^2 + 54x^2}{9} \quad ...(ii)$$

$$\Rightarrow \quad 2x^2(2\pi + 27) = 9K$$

$$\therefore \quad x^2 = \frac{9K}{2(2\pi + 27)} = 3\sqrt{\frac{K}{4\pi + 54}}$$

Now from eq. (i) we have

$$r^2 = \frac{K - 6x^2}{4\pi}$$

$$\Rightarrow \quad r^2 = \frac{\frac{4\pi x^2 + 54x^2}{9} - 6x^2}{4\pi}$$

$$\Rightarrow \quad r^2 = \frac{4\pi x^2 + 54x^2 - 54x^2}{9 \times 4\pi} = \frac{4\pi x^2}{9 \times 4\pi}$$

$$\Rightarrow \quad r^2 = \frac{x^2}{9} \quad \Rightarrow \quad r = \frac{x}{3} \quad \therefore \quad x = 3r$$

Now we have $\dfrac{dV}{dx} = 2x^2 - \dfrac{3x}{\sqrt{\pi}}(K - 6x^2)^{1/2}$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2V}{dx^2} = 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{d}{dx}(K - 6x^2)^{1/2} + (K - 6x^2)^{1/2} \cdot \frac{d}{dx} \cdot x\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[x \cdot \frac{1 \times (-12x)}{2\sqrt{K - 6x^2}} + (K - 6x^2)^{1/2} \cdot 1\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2}{(K - 6x^2)^{1/2}} + (K - 6x^2)^{1/2}\right]$$

$$= 4x - \frac{3}{\sqrt{\pi}}\left[\frac{-6x^2 + K - 6x^2}{(K - 6x^2)^{1/2}}\right] = 4x - \frac{3}{\sqrt{\pi}}\left[\frac{K - 12x^2}{(K - 6x^2)^{1/2}}\right]$$

Put $x = 3 \cdot \sqrt{\dfrac{K}{4\pi + 54}}$

$$\frac{d^2V}{dx^2} = 4 \cdot 3\sqrt{\frac{K}{4\pi + 54}} - \frac{3}{\sqrt{\pi}}\left[\frac{K - 12 \cdot \frac{9K}{4\pi + 54}}{\sqrt{(K - 6 \cdot \frac{9K}{4\pi + 54})}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi + 54}} - \frac{3}{\sqrt{\pi}} \frac{\frac{4K\pi + 54K - 108K}{4\pi + 54}}{\sqrt{\frac{4K\pi + 54K - 54K}{4\pi + 54}}}$$

$$= 12\sqrt{\frac{K}{4\pi + 54}} - \frac{3}{\sqrt{\pi}} \left[\frac{\frac{4K\pi - 54K}{4\pi + 54}}{\sqrt{\frac{4K\pi}{4\pi + 54}}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi + 54}} - \frac{3}{\sqrt{\pi}} \left[\frac{4K\pi - 54K}{\sqrt{4K\pi} \cdot \sqrt{4\pi + 54}}\right]$$

$$= 12\sqrt{\frac{K}{4\pi + 54}} - \frac{6K}{\sqrt{\pi}} \left(\frac{2\pi - 27}{\sqrt{4K\pi} \cdot \sqrt{4\pi + 54}}\right)$$

$$= 12\sqrt{\frac{K}{4\pi + 54}} + \frac{6K}{\sqrt{\pi}} \left[\frac{27 - 2\pi}{\sqrt{4k\pi} \cdot \sqrt{4\pi + 54}}\right] > 0$$

$[\because \quad 27 - 2\pi > 0]$

$\therefore \quad \frac{d^2V}{dx^2} > 0$ so, it is minima.

Hence, the sum of volume is minimum for $x = 3\sqrt{\frac{K}{4\pi + 54}}$

$\therefore$ Minimum volume,

$$\text{V at } \left(x = 3\sqrt{\frac{K}{4\pi + 54}}\right) = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \left(\frac{x}{3}\right)^3$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \cdot \frac{x^3}{27} = \frac{2}{3}x^3 + \frac{4}{81}\pi x^3$$

$$= \frac{2}{3}x^3 \left(1 + \frac{2\pi}{27}\right)$$

Hence, the required minimum volume is $\frac{2}{3}x^3\left(1 + \frac{2\pi}{27}\right)$ and $x = 3r$.

OBJECTIVE TYPE QUESTIONS

Choose the correct answer from the given four options in each of the following questions 35 to 59:

Q35. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is:

(a) $10 \text{ cm}^2/\text{s}$ (b) $\sqrt{3} \text{ cm}^2/\text{s}$

(c) $10\sqrt{3} \text{ cm}^2/\text{s}$ (d) $\frac{10}{3} \text{ cm}^2/\text{s}$

Sol. Let the length of each side of the given equilateral triangle be x cm.

$$\therefore \quad \frac{dx}{dt} = 2 \text{ cm/sec}$$

Area of equilateral triangle $A = \frac{\sqrt{3}}{4}x^2$

$$\therefore \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot 2x \cdot \frac{dx}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{sec}$$

Hence, the rate of increasing of area = $10\sqrt{3} \text{ cm}^2/\text{sec}$.

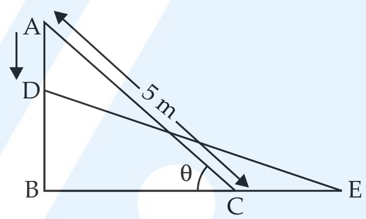
Hence, the correct option is (*c*).

Q36. A ladder, 5 m long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is:

(*a*) $\frac{1}{10}$ radian/sec (*b*) $\frac{1}{20}$ radian/sec

(*c*) 20 radian/sec

(*d*) 10 radian/sec



Sol. Length of ladder = 5 m

Let AB = y m and BC = x m

$\therefore$ In right ΔABC,

$$AB^2 + BC^2 = AC^2$$

$$\Rightarrow \quad x^2 + y^2 = (5)^2 \Rightarrow x^2 + y^2 = 25$$

Differentiating both sides w.r.t x, we have

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad x\frac{dx}{dt} + y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + y \times (-0.1) = 0 \qquad [\because \; x = 2m]$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + \sqrt{25 - x^2} \times (-0.1) = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} + \sqrt{25 - 4} \times (-0.1) = 0$$

$$\Rightarrow \quad 2 \cdot \frac{dx}{dt} - \frac{\sqrt{21}}{10} = 0 \Rightarrow \frac{dx}{dt} = \frac{\sqrt{21}}{20}$$

Now $\cos\theta = \frac{BC}{AC}$ (θ is in radian)

$\Rightarrow$ $\cos\theta = \frac{x}{5}$

Differentiating both sides w.r.t. t, we get

$$\frac{d}{dt}\cos\theta = \frac{1}{5}\cdot\frac{dx}{dt} \Rightarrow -\sin\theta\cdot\frac{d\theta}{dt} = \frac{1}{5}\cdot\frac{\sqrt{21}}{20}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\sqrt{21}}{100}\times\left(-\frac{1}{\sin\theta}\right) = \frac{\sqrt{21}}{100}\times-\left(\frac{1}{\frac{AB}{AC}}\right)$$

$$= -\frac{\sqrt{21}}{100}\times\frac{AC}{AB} = -\frac{\sqrt{21}}{100}\times\frac{5}{\sqrt{21}} = -\frac{1}{20}\text{ radian/sec}$$

[(–) sign shows the decrease of change of angle]

Hence, the required rate = $\frac{1}{20}$ radian/sec

Hence, the correct option is (b).

Q37. The curve $y = x^{1/5}$ has at (0, 0)

(a) a vertical tangent (parallel to y-axis)
(b) a horizontal tangent (parallel to x-axis)
(c) an oblique tangent
(d) no tangent

Sol. Equation of curve is $y = x^{1/5}$

Differentiating w.r.t. x, we get $\frac{dy}{dx} = \frac{1}{5}x^{-4/5}$

(at $x = 0$) $\frac{dy}{dx} = \frac{1}{5}(0)^{-4/5} = \frac{1}{5}\times\frac{1}{0} = \infty$

$$\frac{dy}{dx} = \infty$$

$\therefore$ The tangent is parallel to y-axis.

Hence, the correct option is (a).

Q38. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line $x + 3y = 8$ is

(a) $3x - y = 8$ (b) $3x + y + 8 = 0$
(c) $x + 3y \pm 8 = 0$ (d) $x + 3y = 0$

Sol. Given equation of the curve is $3x^2 - y^2 = 8$ ...(i)

Differentiating both sides w.r.t. x, we get

$$6x - 2y\cdot\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y}$$

$\frac{3x}{y}$ is the slope of the tangent

$\therefore$ Slope of the normal $= \frac{-1}{dy/dx} = \frac{-y}{3x}$

Now $x + 3y = 8$ is parallel to the normal

Differentiating both sides w.r.t. x, we have

$$1 + 3\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{3}$$

$$\therefore \quad \frac{-y}{3x} = -\frac{1}{3} \quad \Rightarrow \quad y = x$$

Putting $y = x$ in eq. (i) we get

$$3x^2 - x^2 = 8 \quad \Rightarrow \quad 2x^2 = 8 \quad \Rightarrow \quad x^2 = 4$$

$$\therefore \quad x = \pm 2 \text{ and } y = \pm 2$$

So the points are (2, 2) and (−2, −2).

Equation of normal to the given curve at (2, 2) is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$$\Rightarrow \quad 3y - 6 = -x + 2 \quad \Rightarrow x + 3y - 8 = 0$$

Equation of normal at (−2, −2) is

$$y + 2 = -\frac{1}{3}(x + 2)$$

$$\Rightarrow \quad 3y + 6 = -x - 2 \quad \Rightarrow x + 3y + 8 = 0$$

$\therefore$ The equations of the normals to the curve are

$$x + 3y \pm 8 = 0$$

Hence, the correct option is (c).

Q39. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of 'a' is:

(a) 1 (b) 0 (c) −6 (d) 6

Sol. Equation of the given curves are $ay + x^2 = 7$ ...(i)

and $x^3 = y$ ...(ii)

Differentiating eq. (i) w.r.t. x, we have

$$a\frac{dy}{dx} + 2x = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{a}$$

$$\therefore \quad m_1 = -\frac{2x}{a} \qquad \left(m_1 = \frac{dy}{dx}\right)$$

Now differentiating eq. (ii) w.r.t. x, we get

$$3x^2 = \frac{dy}{dx} \quad \Rightarrow \quad m_2 = 3x^2 \qquad \left(m_2 = \frac{dy}{dx}\right)$$

The two curves are said to be orthogonal if the angle between the tangents at the point of intersection is 90°.

$\therefore \qquad m_1 \times m_2 = -1$

$\Rightarrow \quad \frac{-2x}{a} \times 3x^2 = -1 \quad \Rightarrow \quad \frac{-6x^3}{a} = -1 \quad \Rightarrow 6x^3 = a$

(1, 1) is the point of intersection of two curves.

$\therefore \qquad 6(1)^3 = a$

So $\qquad a = 6$

Hence, the correct option is (*d*).

Q40. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y?

(*a*) 0.32 (*b*) 0.032 (*c*) 5.68 (*d*) 5.968

Sol. Given that $y = x^4 - 10$

$$\frac{dy}{dx} = 4x^3$$

$$\Delta x = 2.00 - 1.99 = 0.01$$

$$\therefore \qquad \Delta y = \frac{dy}{dx} \cdot \Delta x = 4x^3 \times \Delta x$$

$$= 4 \times (2)^3 \times 0.01 = 32 \times 0.01 = 0.32$$

Hence, the correct option is (*a*).

Q41. The equation of tangent to the curve $y(1 + x^2) = 2 - x$, where it crosses x-axis is:

(*a*) $x + 5y = 2$ (*b*) $x - 5y = 2$

(*c*) $5x - y = 2$ (*d*) $5x + y = 2$

Sol. Given that $y(1 + x^2) = 2 - x$ ...(*i*)

If it cuts x-axis, then y-coordinate is 0.

$\therefore \qquad 0(1 + x^2) = 2 - x \quad \Rightarrow \quad x = 2$

Put $x = 2$ in equation (*i*)

$$y(1 + 4) = 2 - 2 \quad \Rightarrow \quad y(5) = 0 \quad \Rightarrow \quad y = 0$$

Point of contact = (2, 0)

Differentiating eq. (*i*) w.r.t. x, we have

$$y \times 2x + (1 + x^2)\frac{dy}{dx} = -1$$

$$\Rightarrow \quad 2xy + (1 + x^2)\frac{dy}{dx} = -1 \quad \Rightarrow \quad (1 + x^2)\frac{dy}{dx} = -1 - 2xy$$

$$\therefore \quad \frac{dy}{dx} = \frac{-(1 + 2xy)}{(1 + x^2)} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{(2,0)} = \frac{-1}{(1 + 4)} = \frac{-1}{5}$$

Equation of tangent is $y - 0 = -\frac{1}{5}(x - 2)$

$\Rightarrow \qquad 5y = -x + 2 \quad \Rightarrow \quad x + 5y = 2$

Hence, the correct option is (a).