Q19. Show that the line $\frac{x}{a}+\frac{y}{b}=1$, touches the curve $y = b \cdot e^{-x/a}$ at the point where the curve intersects the axis of y.

Sol. Given that $y = b \cdot e^{-x/a}$, the equation of curve

and $\frac{x}{a}+\frac{y}{b}=1$, the equation of line.

Let the coordinates of the point where the curve intersects the y-axis be $(0, y_1)$

Now differentiating $y = b \cdot e^{-x/a}$ both sides w.r.t. x, we get

$$\frac{dy}{dx} = b \cdot e^{-x/a}\left(-\frac{1}{a}\right) = -\frac{b}{a} \cdot e^{-x/a}$$

So, the slope of the tangent, $m_1 = -\frac{b}{a} e^{-x/a}$.

Differentiating $\frac{x}{a}+\frac{y}{b}=1$ both sides w.r.t. x, we get

$$\frac{1}{a}+\frac{1}{b}\cdot\frac{dy}{dx}=0$$

So, the slope of the line, $m_2 = \frac{-b}{a}$.

If the line touches the curve, then $m_1 = m_2$

$\Rightarrow \quad \frac{-b}{a} \cdot e^{-x/a} = \frac{-b}{a} \quad \Rightarrow \quad e^{-x/a} = 1$

$\Rightarrow \quad \frac{-x}{a} \log e = \log 1$ (Taking log on both sides)

$\Rightarrow \quad \frac{-x}{a} = 0 \quad \Rightarrow \quad x = 0$

Putting $x = 0$ in equation $y = b \cdot e^{-x/a}$

$\Rightarrow \quad y = b \cdot e^0 = b$

Hence, the given equation of curve intersect at $(0, b)$ i.e. on y-axis.

Q20. Show that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$ is increasing in **R**.

Sol. Given that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$

Differentiating both sides w.r.t. x, we get

$$f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2}-x} \times \frac{d}{dx}\left(\sqrt{1+x^2}-x\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{\left(\frac{1}{2\sqrt{1+x^2}} \times (2x-1)\right)}{\sqrt{1+x^2}-x}$$

$$= 2 - \frac{1}{1+x^2} + \frac{x - \sqrt{1+x^2}}{\sqrt{1+x^2}\left(\sqrt{1+x^2} - x\right)}$$

$$= 2 - \frac{1}{1+x^2} - \frac{\left(\sqrt{1+x^2} - x\right)}{\sqrt{1+x^2}\left(\sqrt{1+x^2} - x\right)}$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

For increasing function, $f'(x) \geq 0$

$\therefore \quad 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0$

$\Rightarrow \quad \frac{2(1+x^2) - 1 + \sqrt{1+x^2}}{(1+x^2)} \geq 0 \quad \Rightarrow \quad 2 + 2x^2 - 1 + \sqrt{1+x^2} \geq 0$

$\Rightarrow \quad 2x^2 + 1 + \sqrt{1+x^2} \geq 0 \quad \Rightarrow \quad 2x^2 + 1 \geq -\sqrt{1+x^2}$

Squaring both sides, we get $4x^4 + 1 + 4x^2 \geq 1 + x^2$

$\Rightarrow \quad 4x^4 + 4x^2 - x^2 \geq 0 \quad \Rightarrow \quad 4x^4 + 3x^2 \geq 0 \quad \Rightarrow x^2(4x^2 + 3) \geq 0$

which is true for any value of $x \in$ R.

Hence, the given function is an increasing function over R.

Q21. Show that for $a \geq 1$, $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$ is decreasing in **R**.

Sol. Given that: $f(x) = \sqrt{3} \sin x - \cos x - 2ax + b$, $a \geq 1$

Differentiating both sides w.r.t. x, we get

$$f'(x) = \sqrt{3} \cos x + \sin x - 2a$$

For decreasing function, $f'(x) < 0$

$\therefore \quad \sqrt{3} \cos x + \sin x - 2a < 0$

$\Rightarrow \quad 2\left(\frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x\right) - 2a < 0$

$\Rightarrow \quad \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x - a < 0$

$\Rightarrow \quad \left(\cos \frac{\pi}{6} \cos x + \sin \frac{\pi}{6} \sin x\right) - a < 0$

$\Rightarrow \quad \cos\left(x - \frac{\pi}{6}\right) - a < 0$

Since $\cos x \in [-1, 1]$ and $a \geq 1$

$\therefore \quad f'(x) < 0$

Hence, the given function is decreasing in R.