Q11. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square.

Sol. Let area of the first square $A_1 = x^2$
and area of the second square $A_2 = y^2$
Now $A_1 = x^2$ and $A_2 = y^2 = (x - x^2)^2$
Differentiating both A_1 and A_2 w.r.t. t, we get

$$\frac{dA_1}{dt} = 2x \cdot \frac{dx}{dt} \text{ and } \frac{dA_2}{dt} = 2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{\frac{dA_2}{dt}}{\frac{dA_1}{dt}} = \frac{2(x - x^2)(1 - 2x) \cdot \frac{dx}{dt}}{2x \cdot \frac{dx}{dt}}$$

$$= \frac{x(1 - x)(1 - 2x)}{x} = (1 - x)(1 - 2x)$$

$$= 1 - 2x - x + 2x^2 = 2x^2 - 3x + 1$$

Hence, the rate of change of area of the second square with respect to first is $2x^2 - 3x + 1$.

Q12. Find the condition that the curves $2x = y^2$ and $2xy = k$ intersect orthogonally.

Sol. The two circles intersect orthogonally if the angle between the tangents drawn to the two circles at the point of their intersection is 90°.
Equation of the two circles are given as

$$2x = y^2 \quad \text{...(i)}$$

and $$2xy = k \quad \text{...(ii)}$$

Differentiating eq. (*i*) and (*ii*) w.r.t. x, we get

$$2.1 = 2y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{y} \Rightarrow m_1 = \frac{1}{y}$$

(m_1 = slope of the tangent)

$$\Rightarrow \quad 2xy = k$$

$$\Rightarrow \quad 2\left[x \cdot \frac{dy}{dx} + y \cdot 1\right] = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_2 = -\frac{y}{x}$$

[m_2 = slope of the other tangent]

If the two tangents are perpendicular to each other,
then $m_1 \times m_2 = -1$

$$\Rightarrow \quad \frac{1}{y} \times \left(-\frac{y}{x}\right) = -1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

Now solving $2x = y^2$ [From (*i*)]

and $2xy = k$ [From (*ii*)]

From eq. (*ii*) $y = \frac{k}{2x}$

Putting the value of y in eq. (*i*)

$$2x = \left(\frac{k}{2x}\right)^2 \Rightarrow 2x = \frac{k^2}{4x^2}$$

$$\Rightarrow \quad 8x^3 = k^2 \Rightarrow 8(1)^3 = k^2 \Rightarrow 8 = k^2$$

Hence, the required condition is $k^2 = 8$.

Q13. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

Sol. Given circles are $xy = 4$ ...(*i*)

and $x^2 + y^2 = 8$ ...(*ii*)

Differentiating eq. (*i*) w.r.t., x

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_1 = -\frac{y}{x} \quad \text{...(iii)}$$

where, m_1 is the slope of the tangent to the curve.

Differentiating eq. (*ii*) w.r.t. x

$$2x + 2y \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow m_2 = -\frac{x}{y}$$

where, m_2 is the slope of the tangent to the circle.

To find the point of contact of the two circles

$$m_1 = m_2 \Rightarrow -\frac{y}{x} = -\frac{x}{y} \Rightarrow x^2 = y^2$$

Putting the value of y^2 in eq. (*ii*)

$$x^2 + x^2 = 8 \Rightarrow 2x^2 = 8 \Rightarrow x^2 = 4$$

$$\therefore \quad x = \pm 2$$

$$\because \quad x^2 = y^2 \Rightarrow y = \pm 2$$

$\therefore$ The point of contact of the two circles are (2, 2) and (−2, 2).

Q14. Find the coordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.

Sol. Equation of curve is given by $\sqrt{x} + \sqrt{y} = 4$

Let (x_1, y_1) be the required point on the curve

$$\therefore \quad \sqrt{x_1} + \sqrt{y_1} = 4$$

Differentiating both sides w.r.t. x_1, we get

$$\frac{d}{dx_1}\sqrt{x_1} + \frac{d}{dx_1}\sqrt{y_1} = \frac{d}{dx_1}(4)$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x_1}} + \frac{1}{2\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{y_1}} \cdot \frac{dy_1}{dx_1} = 0 \Rightarrow \frac{dy_1}{dx_1} = -\frac{\sqrt{y_1}}{\sqrt{x_1}} \quad ...(i)$$

Since the tangent to the given curve at (x_1, y_1) is equally inclined to the axes.

$\therefore$ Slope of the tangent $\frac{dy_1}{dx_1} = \pm \tan \frac{\pi}{4} = \pm 1$

So, from eq. (*i*) we get

$$-\frac{\sqrt{y_1}}{\sqrt{x_1}} = \pm 1$$

Squaring both sides, we get

$$\frac{y_1}{x_1} = 1 \quad \Rightarrow \quad y_1 = x_1$$

Putting the value of y_1 in the given equation of the curve.

$$\sqrt{x_1} + \sqrt{y_1} = 4$$

$$\Rightarrow \quad \sqrt{x_1} + \sqrt{x_1} = 4 \quad \Rightarrow \quad 2\sqrt{x_1} = 4 \quad \Rightarrow \quad \sqrt{x_1} = 2 \quad \Rightarrow \quad x_1 = 4$$

Since $\quad y_1 = x_1$

$\therefore \quad y_1 = 4$

Hence, the required point is (4, 4).

Q15. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

Sol. We know that the angle of intersection of two curves is equal to the angle between the tangents drawn to the curves at their point of intersection.

The given curves are $y = 4 - x^2$ ... (*i*) and $y = x^2$ ...(*ii*)

Differentiating eq. (*i*) and (*ii*) with respect to x, we have

$$\frac{dy}{dx} = -2x \quad \Rightarrow \quad m_1 = -2x$$

m_1 is the slope of the tangent to the curve (*i*).

and $\quad \frac{dy}{dx} = 2x \quad \Rightarrow \quad m_2 = 2x$

m_2 is the slope of the tangent to the curve (*ii*).

So, $\quad m_1 = -2x$ and $m_2 = 2x$

Now solving eq. (*i*) and (*ii*) we get

$$\Rightarrow \quad 4 - x^2 = x^2 \quad \Rightarrow \quad 2x^2 = 4 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow \quad x = \pm\sqrt{2}$$

So, $\quad m_1 = -2x = -2\sqrt{2}$ and $m_2 = 2x = 2\sqrt{2}$

Let θ be the angle of intersection of two curves

$$\therefore \quad \tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$= \left|\frac{2\sqrt{2} + 2\sqrt{2}}{1 - (2\sqrt{2})(2\sqrt{2})}\right| = \left|\frac{4\sqrt{2}}{1-8}\right| = \left|\frac{4\sqrt{2}}{-7}\right| = \frac{4\sqrt{2}}{7}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$$

Hence, the required angle is $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$.

Q16. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

Sol. Given that the equation of the two curves are $y^2 = 4x$ ...(*i*)

and $x^2 + y^2 - 6x + 1 = 0$ ...(*ii*)

Differentiating (*i*) w.r.t. x, we get $2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

Slope of the tangent at (1, 2), $m_1 = \frac{2}{2} = 1$

Differentiating (*ii*) w.r.t. $x \Rightarrow 2x + 2y \cdot \frac{dy}{dx} - 6 = 0$

$$\Rightarrow \quad 2y \cdot \frac{dy}{dx} = 6 - 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{6-2x}{2y}$$

$\therefore$ Slope of the tangent at the same point (1, 2)

$$\Rightarrow \quad m_2 = \frac{6 - 2\times 1}{2\times 2} = \frac{4}{4} = 1$$

We see that $m_1 = m_2 = 1$ at the point (1, 2).

Hence, the given circles touch each other at the same point (1, 2).

Q17. Find the equation of the normal lines to the curve $3x^2 - y^2 = 8$ which are parallel to the line $x + 3y = 4$.

Sol. We have equation of the curve $3x^2 - y^2 = 8$

Differentiating both sides w.r.t. x, we get

$$\Rightarrow \quad 6x - 2y\cdot\frac{dy}{dx} = 0 \quad \Rightarrow \quad -2y\frac{dy}{dx} = -6x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3x}{y}$$

Slope of the tangent to the given curve $= \frac{3x}{y}$

$\therefore$ Slope of the normal to the curve $= -\frac{1}{\frac{3x}{y}} = -\frac{y}{3x}$.

Now differentiating both sides the given line $x + 3y = 4$

$\Rightarrow \quad 1 + 3 \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{3}$

Since the normal to the curve is parallel to the given line $x + 3y = 4$.

$\therefore \quad -\frac{y}{3x} = -\frac{1}{3} \quad \Rightarrow \quad y = x$

Putting the value of y in $3x^2 - y^2 = 8$, we get

$$3x^2 - x^2 = 8 \quad \Rightarrow 2x^2 = 8 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$$

$\therefore \quad y = \pm 2$

$\therefore$ The points on the curve are (2, 2) and (– 2, – 2).

Now equation of the normal to the curve at (2, 2) is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$\Rightarrow \quad 3y - 6 = -x + 2 \quad \Rightarrow x + 3y = 8$

at (– 2, – 2) $\quad y + 2 = -\frac{1}{3}(x + 2)$

$\Rightarrow \quad 3y + 6 = -x - 2 \quad \Rightarrow x + 3y = -8$

Hence, the required equations are $x + 3y = 8$ and $x + 3y = -8$ or $x + 3y = \pm 8$.

Q18. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?

Sol. Given that the equation of the curve is

$$x^2 + y^2 - 2x - 4y + 1 = 0 \quad ...(i)$$

Differentiating both sides w.r.t. x, we have

$$2x + 2y \cdot \frac{dy}{dx} - 2 - 4 \cdot \frac{dy}{dx} = 0$$

$\Rightarrow \quad (2y - 4)\frac{dy}{dx} = 2 - 2x \Rightarrow \frac{dy}{dx} = \frac{2 - 2x}{2y - 4} \quad ...(ii)$

Since the tangent to the curve is parallel to the y-axis.

$\therefore \quad \text{Slope } \frac{dy}{dx} = \tan\frac{\pi}{2} = \infty = \frac{1}{0}$

So, from eq. (*ii*) we get

$$\frac{2 - 2x}{2y - 4} = \frac{1}{0} \quad \Rightarrow \quad 2y - 4 = 0 \quad \Rightarrow \quad y = 2$$

Now putting the value of y in eq. (*i*), we get

$\Rightarrow \quad x^2 + (2)^2 - 2x - 8 + 1 = 0$

$\Rightarrow \quad x^2 - 2x + 4 - 8 + 1 = 0$

$\Rightarrow \quad x^2 - 2x - 3 = 0 \quad \Rightarrow \quad x^2 - 3x + x - 3 = 0$

$\Rightarrow \quad x(x - 3) + 1(x - 3) = 0 \quad \Rightarrow (x - 3)(x + 1) = 0$

$\Rightarrow \quad x = -1 \text{ or } 3$

Hence, the required points are (– 1, 2) and (3, 2).