

Properties of Matrix Multiplication

1. Matrix Multiplication is associative, i.e., for any 3 matrices A, B & C such that A & B are compatible for multiplying and B & C are compatible for multiplying, then $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

2. Matrix Multiplication is non commutative in general.

Example: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ & $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix} \therefore AB \neq BA$$

This holds true even for square matrices of same order.

3. If α & β are any two scalars such that $\alpha \cdot \beta = 0$, then either $\alpha = 0$ or $\beta = 0$.

This need not be true for matrices.

Example: Let $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ the zero matrix.}$$

Elementary row operations of row echelon matrices.

Definition: A matrix is called row reduced echelon if the following properties hold:

- i) Every zero row is below every non-zero row.
- ii) The leading coefficient (first non-zero coefficient) of every row is 1.
- iii) A column which contains the leading coefficient 1 of a row has all the other coefficients equal to zero.
- iv) Suppose the matrix has r non-zero rows. If the leading non-zero entry of the i^{th} row occurs in the k_i^{th} column, then

$$k_1 < k_2 < \dots < k_r.$$

Row elementary operations

- 1) Multiplying the i^{th} row by a non-zero scalar, say λ .

$$R_i \rightarrow \lambda R_i.$$

- 2) Interchanging i^{th} row & j^{th} column

$$R_i \leftrightarrow R_j$$

- 3) Replace the i^{th} row by the sum of i^{th} row & μ -multiple of j^{th} row.

$$R_i \rightarrow R_i + \mu R_j$$

Procedure to obtain a row reduced echelon matrix from a given matrix

- Step 1: Apply interchange of rows to push down the zero rows to the end of the matrix.
- Step 2: Find the first non-zero column (from left). (let us suppose that it is k .)
- Step 3: Again apply interchange of rows to push up a row whose leading non-zero coefficient occurs in first non-zero column to the first row. Divide the first row by the leading non-zero coefficient, so that the leading non-zero coefficient becomes 1.
- Step 4: Next apply $R_i \mapsto R_i + \mu R_1$ for suitable values of i & μ so that the first non-zero column has non-zero coefficients only in the first row.
- Step 5: Repeat steps 2 to 4. for the submatrix obtained by deleting the 1st row & 1st column until all the non-zero rows are exhausted.